

DYADIC-PROBABILISTIC METHODS IN BILINEAR ANALYSIS

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ABSTRACT. We demonstrate and develop dyadic-probabilistic methods in connection with non-homogeneous bilinear operators, namely singular integrals and square functions. In doing so we also advance the linear theory of Calderón–Zygmund operators by improving techniques and results. Our main tools include maximal truncations, adapted Cotlar type inequalities and suppression and big piece methods.

1. INTRODUCTION

The best known boundedness results for usual Calderón–Zygmund operators and square functions are proved using dyadic analysis and probabilistic methods. The general philosophy was originally introduced by Nazarov–Treil–Volberg to deal with non-homogeneous measures, see e.g. [23], [21]. Such techniques have then been widely used and refined to multiple directions. See e.g. Azzam–Hofmann–Martell–Mayboroda–Mourgoglou–Tolsa–Volberg [1] (rectifiability of the harmonic measure), Hytönen [7] (A_2 theorem and the representation of singular integrals), Lacey–Sawyer–Uriarte-Tuero–Shen [12] and Lacey [11] (two weight inequality for the Hilbert transform), Lacey–Martikainen [13] (non-homogeneous local Tb theorem with L^2 testing conditions), Martikainen [15] (representation of bi-parameter singular integrals), and Tolsa [24], [25] (general account of non-homogeneous theory and the Painlevé’s problem).

We study the dyadic martingale structure behind **bilinear** operators, mainly bilinear Calderón–Zygmund and square function operators, in the setting of non-homogeneous analysis. However, we also continue to push some of the most recent and advanced techniques further. Thus, our proof techniques also yield some new insight about the basic linear theory, for example, by replacing certain previous methods by the wider use of maximal truncations, various Cotlar type inequalities, and the big piece and suppression methods. These four aspects

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really are the cornerstone to our approach, and we shall carefully explain their place in the proof when we encounter them.

Firstly, we take a look at what dyadic–probabilistic methods can do, and how to use them, in the multilinear world. It appears that these methods have not really been used before in this setting. This entails recording the general probabilistic martingale proof structure for bilinear operators. It also turns out that the treatment of non-homogeneous singular integrals is quite involved in this setting with relevant technical challenges for example at the diagonal part. Secondly, we exploit and develop certain very recent techniques. In particular, the big pieces method – see e.g. Martikainen–Mourgoglou–Tolsa [18] and Martikainen–Mourgoglou–Vuorinen [19] – is extended to bilinear operators and is used to improve some integrability assumptions. In fact, we prove a global Tb theorem with a new proof and weaker testing conditions than usual (we can go below L^1). In particular, we don’t use the $\text{RBMO}(\mu)$ space in our proof at all (as e.g. in [23]).

We consider the following philosophy, which we also follow in this paper, to be the most important new realisations in the Tb world: testing conditions involving the maximal truncations T_\sharp , instead of the original Calderón–Zygmund operator T , are much easier to exploit (via suppression methods). This is the case even if the testing conditions involving T_\sharp are extremely weak. This idea seems to originate from the paper by Nazarov, Treil and Volberg [22], where they prove a special big piece type Tb theorem for Cauchy integral type operators in connection with Vitushkin’s conjecture. The full potential of this approach was not immediately used in the Tb circles, rather it has really started to become clear only recently – see e.g. Hytönen–Nazarov [9] and Martikainen–Mourgoglou–Tolsa [18]. One of the fundamental problems is that we only want to assume conditions involving T itself, and the passage to conditions involving T_\sharp can be tricky. This requires some kind of adapted Cotlar’s inequality i.e. Cotlar’s inequality which only uses the assumed testing conditions instead of some form of a priori boundedness. There are no such problems in the context of square functions, which explains, in part, why square functions are so much simpler to handle.

Let us get back to the multilinear theory. The literature on multilinear analysis is certainly vast. We mention just some closely related papers here. Recent papers concerning multilinear $T1$ or Tb type theorems are e.g. Grafakos–Oliveira [2], J. Hart [4, 5, 6] and Kovač–Thiele [10]. Some formulations related to multilinear *local* Tb theorems appear in Grau de la Herrán–Hart–Oliveira [3] and Mirek–Thiele [20], but the multilinear local theory seems to have various restrictions which require further understanding. This is one of our motivations also, but we refrain from touching that part of the theory too much in this paper. Indeed, things already get quite technical.

We now formulate the setting and our main theorem. Of course, many of the needed results, the proofs, and the big picture of the proof are as interesting as the main theorem. We will lay down the structure of the proof later.

A function

$$K: (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta \rightarrow \mathbb{C}, \quad \Delta := \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n: x = y = z\},$$

is called a standard bilinear m -dimensional Calderón–Zygmund kernel if for some $\alpha \in (0, 1]$ and $C_K < \infty$ it holds that

$$|K(x, y, z)| \leq \frac{C_K}{(|x - y| + |x - z|)^{2m}},$$

$$|K(x, y, z) - K(x', y, z)| \leq C_K \frac{|x - x'|^\alpha}{(|x - y| + |x - z|)^{2m+\alpha}}$$

whenever $|x - x'| \leq \max(|x - y|, |x - z|)/2$,

$$|K(x, y, z) - K(x, y', z)| \leq C_K \frac{|y - y'|^\alpha}{(|x - y| + |x - z|)^{2m+\alpha}}$$

whenever $|y - y'| \leq \max(|x - y|, |x - z|)/2$, and

$$|K(x, y, z) - K(x, y, z')| \leq C_K \frac{|z - z'|^\alpha}{(|x - y| + |x - z|)^{2m+\alpha}}$$

whenever $|z - z'| \leq \max(|x - y|, |x - z|)/2$.

Given two Radon measures ν_1, ν_2 on \mathbb{R}^n , possibly complex, we define, whenever the right hand side makes sense, that

$$T_\varepsilon(\nu_1, \nu_2)(x) = \iint_{\max(|x-y|, |x-z|) > \varepsilon} K(x, y, z) d\nu_1(y) d\nu_2(z), \quad x \in \mathbb{R}^n, \varepsilon > 0.$$

The defining integral is absolutely convergent e.g. if $|\nu_i|(\mathbb{R}^n) < \infty$ for $i = 1, 2$. The truncations could also be defined as

$$\tilde{T}_\varepsilon(\nu_1, \nu_2)(x) = \iint_{|x-y|^2 + |x-z|^2 > \varepsilon^2} K(x, y, z) d\nu_1(y) d\nu_2(z), \quad x \in \mathbb{R}^n, \varepsilon > 0.$$

We prefer T_ε over \tilde{T}_ε as it seems somewhat easier to work with in connection with some pointwise estimates. However, our main theorem, Theorem 1.2, can also be stated using \tilde{T}_ε . In fact, such a version follows from the one with T_ε using that

$$|T_\varepsilon(\nu_1, \nu_2)(x) - \tilde{T}_\varepsilon(\nu_1, \nu_2)(x)| \lesssim M_m \nu_1(x) M_m \nu_2(x),$$

where

$$M_m \nu(x) = \sup_{r>0} \frac{|\nu|(B(x, r))}{r^m}, \quad x \in \mathbb{R}^n.$$

For us a bilinear m -dimensional SIO (singular integral operator) T is simply the collection $(T_\varepsilon)_{\varepsilon>0}$ in the sense that we are only interested in some uniform in $\varepsilon > 0$ boundedness properties of the operators T_ε . This is all simply determined by the given kernel K .

We continue to define the maximal truncations as follows:

$$\begin{aligned} T_{\sharp,\delta}(\nu_1, \nu_2)(x) &= \sup_{\varepsilon > \delta} |T_\varepsilon(\nu_1, \nu_2)(x)|, \quad \delta \geq 0; \\ T_{\sharp}(\nu_1, \nu_2)(x) &= T_{\sharp,0}(\nu_1, \nu_2)(x). \end{aligned}$$

A positive Radon measure μ on \mathbb{R}^n is said to be of order m if for some constant $C_\mu < \infty$ we have $\mu(B(x, r)) \leq C_\mu r^m$ for all $x \in \mathbb{R}^n$ and $r > 0$. We set

$$T_{\mu,\varepsilon}(f, g)(x) = T_\varepsilon(f d\mu, g d\mu)(x) = \iint_{\max(|x-y|, |x-z|) > \varepsilon} K(x, y, z) f(y) g(z) d\mu(y) d\mu(z).$$

The above is well-defined as an absolutely convergent integral if e.g. $f \in L^{p_1}(\mu)$ and $g \in L^{p_2}(\mu)$ for some $p_1, p_2 \in [1, \infty)$, since then

$$(1.1) \quad \iint_{\max(|x-y|, |x-z|) > \varepsilon} |K(x, y, z) f(y) g(z)| d\mu(y) d\mu(z) \lesssim \frac{1}{\varepsilon^{m(1/p_1+1/p_2)}} \|f\|_{L^{p_1}(\mu)} \|g\|_{L^{p_2}(\mu)}.$$

We also set

$$\begin{aligned} T_{\mu,\sharp,\delta}(f, g)(x) &= \sup_{\varepsilon > \delta} |T_{\mu,\varepsilon}(f, g)(x)|, \quad \delta \geq 0; \\ T_{\mu,\sharp}(f, g)(x) &= T_{\mu,\sharp,0}(f, g)(x). \end{aligned}$$

The notation T^{1*} and T^{2*} stand for the adjoints of a bilinear operator T , i.e.

$$\langle T(f, g), h \rangle = \langle T^{1*}(h, g), f \rangle = \langle T^{2*}(f, h), g \rangle.$$

To state the main theorem we still need the concept of doubling cubes and cubes with small boundary. See also the end of the introduction for additional notation, which is rather standard. A cube $Q \subset \mathbb{R}^n$ is called (α, β) -doubling for a given Radon measure μ if $\mu(\alpha Q) \leq \beta \mu(Q)$. Given $t > 0$ we say that a cube $Q \subset \mathbb{R}^n$ has t -small boundary with respect to the measure μ if

$$\mu(\{x \in 2Q : \text{dist}(x, \partial Q) \leq \lambda \ell(Q)\}) \leq t \lambda \mu(2Q)$$

for every $\lambda > 0$. The definition of the suppressed operators T_Φ can be found in Section 3. They appear in the formulation of the theorem in connection with a weak boundedness property – at this point one should simply understand that it is a purely diagonal condition, a necessary condition (as we will show), and is automatically satisfied should K possess some antisymmetry.

1.2. Theorem. *Let μ be a measure of order m on \mathbb{R}^n and T be a bilinear m -dimensional SIO. Let b and t be large enough dimensional constants, $s, c_b > 0$ and $C_b, C_W, C_{\text{test}} < \infty$. Let the functions b_i , $i = 1, 2, 3$, be such that $\|b_i\|_{L^\infty(\mu)} \leq C_b$ and*

$$|\langle b_i \rangle_Q^\mu| \geq c_b \quad \text{for all cubes } Q.$$

We assume the weak boundedness property in the form that

$$\sup_{\delta > 0} |\langle T_{\mu,\Phi,\delta}(1_Q b_1, 1_Q b_2), 1_Q b_3 \rangle_\mu| \leq C_W \mu(5Q) \quad \text{for all cubes } Q$$

whenever $\Phi: \mathbb{R}^n \rightarrow [0, \infty)$ is a 1-Lipschitz function. Suppose that for every $(2, b)$ -doubling cube Q with t -small boundary we have the testing condition

$$\sup_{\delta > 0} \sup_{\lambda > 0} \lambda^s \mu(\{x \in Q: |S_{\mu, \delta}(b1_Q, b'1_Q)(x)| > \lambda\}) \leq C_{\text{test}} \mu(Q)$$

for all the choices $(S, b, b') \in \{(T, b_1, b_2), (T^{1*}, b_3, b_2), (T^{2*}, b_1, b_3)\}$. Then for all $1 < p, q < \infty$ and $1/2 < r < \infty$ satisfying $1/p + 1/q = 1/r$ we have that

$$\|T_{\mu, \sharp}\|_{L^p(\mu) \times L^q(\mu) \rightarrow L^r(\mu)} \lesssim 1$$

with a constant depending on p, q, r , the above fixed constants, and the constants appearing in the definitions of T and μ .

The proof does not involve any type of bilinear interpolation. Rather, it has the following steps:

- (1) *Prove testing conditions for T_{\sharp} .* This is Corollary 2.3 and it requires the adapted Cotlar's inequality Proposition 2.1. This step is motivated by the techniques used in the very recent proof of a certain (linear) local Tb theorem by Martikainen–Mourgoglou–Tolsa [18]. See also Hytönen–Nazarov [9] for the first instance of such an adapted Cotlar's inequality in the Lebesgue case.
- (2) *Introduce suppressed operators T_{Φ} .* This is done in Section 3. These originate (in some special case) from [22], and have since then been used quite a lot. The idea is that if Φ is chosen suitably, then T_{Φ} behaves better than T , but also agrees with T on the set $\{\Phi = 0\}$ – which is arranged to be relatively large. This is called suppression. These operators are needed in the next step – the big piece Tb . The bilinear suppression details do not differ too much from the linear ones, but we believe our presentation should be logical and nicely readable.

The following technical thing needs to be noted. The suppressed operators already make an appearance in the statement of the main theorem, because we use a somewhat non-classical formulation of the weak boundedness property. However, it is still a purely diagonal condition, a necessary condition (as we will show), and is automatically satisfied should K possess some antisymmetry. We seem to need this since our proof strategy goes through a new formulation of the big piece Tb . The upshot is that we can allow $s < 1$ in the main testing conditions.

- (3) *Prove a version of the big piece Tb that can be applied to prove the main theorem.* This is Theorem 4.2. The formulation is necessarily relatively technical, and cannot be fully described here. Briefly, the testing conditions for T_{\sharp} (proved in step 1) allow us to do the suppression from Step 2. The big piece Tb allows us to conclude that the operators T_{ε} are, uniformly in $\varepsilon > 0$, bounded on a big piece of a given nice cube Q (of small boundary).

The proof of this contains the bilinear dyadic–probabilistic Tb argument, and so forms the technical core of the paper. There are many details here to be noted. The bilinear framework seems to complicate at

least the treatment of the non-homogeneous paraproduct and the diagonal. We also simplify quite a few details from the linear theory with some new summation arguments – we, for example, can make do without some standard matrix summation lemmas previously extensively used in these arguments.

- (4) *Prove weak type end point estimates for T_{\sharp} .* This is done in Section 5, and is laborious and technical in our generality. We chose to give the full details. This part is separate from the rest of the steps as it is essentially basic theory of non-homogeneous bilinear Calderón–Zygmund operators – but we need to write it here since we are unaware of references operating in our generality. A more standard formulation (than in step 1) of Cotlar’s inequality is also needed here. The conclusions need to involve T_{\sharp} , since the good lambda method (step 6) requires it.
- (5) *Prove a bilinear adaptation of the good lambda method of Tolsa.* This is Theorem 6.3. It is to be noted that the bilinear version is very straightforward to prove mimicking the linear proof.
- (6) *Synthesis.* In Section 7 we give the proof of the main theorem, Theorem 1.2. It is a very short argument using the steps 1-6. It is probably instructive to take a look at this proof to get an idea of the big picture before looking at all the details.

In addition to the above, we demonstrate and make very heavy use of various dyadic L^p , $p \neq 2$, techniques. This is because we choose to prove our big piece Tb , Theorem 4.2, directly with general exponents $1 < p, q, r < \infty$. For the proof of our main theorem, we only need this big piece Tb with $r = 2$ and $p = q = 4$. It is possible that the proof could be somewhat simpler in this case, but the space L^4 would appear anyway. We prefer the general L^p techniques.

Let us mention that we briefly discuss the much simpler case of square functions in Section 8.

Additional notation. We write $A \lesssim B$, if there is a constant $C > 0$ (depending only on some fixed constants like m, n, α etc.) so that $A \leq CB$. Moreover, $A \lesssim_{\tau} B$ means that the constant C can also depend on some relevant given parameter $\tau > 0$. We may also write $A \sim B$ if $B \lesssim A \lesssim B$.

We then define some notation related to cubes. If Q and R are two cubes we set:

- $\ell(Q)$ is the side-length of Q ;
- If $a > 0$, we denote by aQ the cube that is concentric with Q and has sidelength $a\ell(Q)$;
- $d(Q, R) = \text{dist}(Q, R)$ denotes the distance between the cubes Q and R ;
- $D(Q, R) := d(Q, R) + \ell(Q) + \ell(R)$ is the long distance;
- $\text{ch}(Q)$ denotes the dyadic children of Q ;
- $\mu|_Q$ denotes the measure μ restricted to Q ;

- If Q is in a dyadic grid, then $Q^{(k)}$ denotes the unique dyadic cube S in the same grid so that $Q \subset S$ and $\ell(S) = 2^k \ell(Q)$;
- If \mathcal{D} is a dyadic grid, then $\mathcal{D}_k = \{Q \in \mathcal{D} : \ell(Q) = 2^{-k}\}$;
- $\langle f \rangle_Q^\mu = \mu(Q)^{-1} \int_Q f d\mu = \langle f \rangle_Q$ (when the measure is clear from the context). One can interpret this to equal zero if $\mu(Q) = 0$.

The notation $\langle f, g \rangle_\mu$ stands for the pairing $\int f g d\mu$.

The following maximal functions are also used:

$$\begin{aligned}
M_{\mu, \mathcal{D}} f(x) &= \sup_{Q \in \mathcal{D}} 1_Q(x) \langle |f| \rangle_Q^\mu \quad (\mathcal{D} \text{ is a dyadic grid}); \\
M_{\mu, m} f(x) &= \sup_{r>0} \frac{1}{r^m} \int_{B(x, r)} |f| d\mu; \\
M_\mu f(x) &= \sup_{r>0} \langle |f| \rangle_{B(x, r)}^\mu; \\
M_\mu^Q f(x) &= \sup_{r>0} \langle |f| \rangle_{Q(x, r)}^\mu; \\
N_\mu f(x) &= \sup \left\{ \frac{1}{\mu(5B)} \int_B |f| d\mu : B \text{ is a ball containing the point } x \right\}.
\end{aligned}$$

Here $Q(x, r)$ stands for the open cube with center x and side length $2r$, while $B(x, r) = \{y : |x - y| < r\}$. Given $s > 0$ we define the s -adapted maximal functions as in $M_{\mu, s} f(x) = M_\mu(|f|^s)(x)^{1/s}$. The bilinear variants are defined in the natural way, e.g.

$$M_\mu(f, g)(x) = \sup_{r>0} \langle |f| \rangle_{B(x, r)}^\mu \langle |g| \rangle_{B(x, r)}^\mu.$$

We can also hit complex measures ν with these maximal functions – simply replace the appearing integrals $\int_A |f| d\mu$ with $|\nu|(A)$, e.g. $M_\mu \nu(x) = \sup_{r>0} \frac{|\nu|(B(x, r))}{\mu(B(x, r))}$.

The following additional notation for singular integrals is occasionally useful. We want to sometimes be able to e.g. hit $f \otimes g(y, z) := f(y)g(z)$ instead of the pair (f, g) – to enable this we use the notation \tilde{T} as below. For every $\varepsilon > 0$ and measure σ in \mathbb{R}^{2n} we formally define $\tilde{T}_\varepsilon \sigma$ by setting

$$\tilde{T}_\varepsilon \sigma(x) := \iint_{\max(|x-y|, |x-z|) > \varepsilon} K(x, y, z) d\sigma(y, z), \quad x \in \mathbb{R}^n.$$

All the other notions involving \tilde{T} are defined analogously.

Lastly, we record here the following standard estimate that we shall have frequent use for.

1.3. Lemma. *Let $x \in \mathbb{R}^n$ and $t > 0$. The following estimate holds*

$$\iint \frac{d|\nu_1|(y) d|\nu_2|(z)}{(t + |x - y| + |x - z|)^{2m+\alpha}} \lesssim t^{-\alpha} M_m(\nu_1, \nu_2)(x)$$

for all appropriate complex measures ν_1 and ν_2 .

Proof. Simply split the domain of integration to $\max(|x - y|, |x - z|) < t$ and $2^{k-1}t \leq \max(|x - y|, |x - z|) < 2^k t$, $k \geq 1$, and estimate in a straightforward way. \square

2. COTLAR TYPE INEQUALITY AND TESTING CONDITION FOR $T_{\mu, \sharp}$

The purpose of this section is to show that uniform testing conditions concerning $T_{\mu, \delta}(1_Q b_1, 1_Q b_2)$, $\delta > 0$, imply testing conditions for $T_{\mu, \sharp}(1_Q b_1, 1_Q b_2)$. This is important when we want to apply the big pieces type Tb theorem. Outside an arbitrarily small set we achieve this improved testing via the following version of Cotlar's inequality. It is extremely important to note that this version of Cotlar only uses the assumed testing conditions – not some form of a priori boundedness.

2.1. Proposition. *Let μ be a measure of degree m on \mathbb{R}^n , T be a bilinear m -dimensional SIO and $s, \delta > 0$. Let b and t be large enough constants depending only on the dimension n . Suppose $b_i \in L^\infty(\mu)$, $i = 1, 2$, satisfy for every $(2, b)$ -doubling cube R with t -small boundary that*

$$\sup_{\lambda > 0} \lambda^s \mu(\{x \in R: |T_{\mu, \delta}(1_R b_1, 1_R b_2)(x)| > \lambda\}) \lesssim \mu(R).$$

Suppose $Q \subset \mathbb{R}^n$ is a fixed cube and $\tau > 0$. Then uniformly for every $\varepsilon > \delta$ and $x \in (1 - \tau)Q$ there holds that

$$|T_{\mu, \varepsilon}(1_Q b_1, 1_Q b_2)(x)| \lesssim C(\tau) + M_{\mu, s/4}^Q(1_Q T_{\mu, \delta}(1_Q b_1, 1_Q b_2))(x).$$

Proof. Fix $x \in (1 - \tau)Q$ and $\varepsilon_0 > \delta$. Let $C(n)$ be a large dimensional constant. In what follows we will implicitly need that b is sufficiently much larger than $C(n)$, say $b > C(n)^{n+1} \geq C(n)^{m+1}$. Choose the smallest k so that $B(x, C(n)^k \varepsilon_0)$ is $((C(n), b)$ -doubling. Set $\varepsilon = C(n)^k \varepsilon_0$. Notice that

$$|T_{\mu, \varepsilon_0}(1_Q b_1, 1_Q b_2)(x) - T_{\mu, \varepsilon}(1_Q b_1, 1_Q b_2)(x)| \lesssim \int_{\varepsilon_0 < |x-w| \leq \varepsilon} \frac{d\mu(w)}{|w-x|^m} \lesssim 1.$$

The last estimate is seen using a standard calculation based on the choice of ε . This calculation is performed carefully in somewhat more generality at the beginning of the proof of Proposition 5.15. Therefore, it suffices to estimate $|T_{\mu, \varepsilon}(1_Q b_1, 1_Q b_2)(x)|$.

If ε happens to be large enough compared to $\ell(Q)$, this is easy. Indeed, for $\varepsilon > c_\tau \ell(Q)$, say, we have

$$|T_{\mu, \varepsilon}(1_Q b_1, 1_Q b_2)(x)| \lesssim \mu(Q) \int \frac{d\mu(z)}{(\varepsilon + |x-z|)^{2m}} \lesssim_\tau \frac{\mu(Q)}{\ell(Q)^m} \lesssim 1.$$

We can therefore assume that $\varepsilon \leq c_\tau \ell(Q)$ for a sufficiently small constant $c_\tau > 0$. Choose (using Lemma 9.43 of [24]) a cube R centred at x so that it has t -small boundary with respect to μ , and

$$B(x, \varepsilon) \subset R \subset B(x, C_n \varepsilon) \subset Q.$$

The last inclusion holds if c_τ is fixed small enough. Notice that R is $(2, b)$ -doubling as

$$\mu(2R) \leq \mu(B(x, C(n)\varepsilon)) \leq b\mu(B(x, \varepsilon)) \leq b\mu(R),$$

where we used that $C(n)$ was chosen large in the beginning. In particular, we have for some $C_0 < \infty$ that

$$(2.2) \quad \mu(\{w \in R: |T_{\mu, \delta}(1_R b_1, 1_R b_2)(w)| > \lambda\}) \leq C_0 \frac{\mu(R)}{\lambda^s}$$

for every $\lambda > 0$.

Write for fixed $w \in R$ the equality

$$\begin{aligned} T_{\mu, \varepsilon}(1_Q b_1, 1_Q b_2)(x) &= T_{\mu, \varepsilon}(1_Q b_1, 1_Q b_2)(x) - T_{\mu, \delta}(1_{(2R)^c} 1_Q b_1, 1_Q b_2)(w) \\ &\quad + T_{\mu, \delta}(1_Q b_1, 1_Q b_2)(w) - T_{\mu, \delta}(1_{2R} 1_Q b_1, 1_Q b_2)(w). \end{aligned}$$

Notice that $(2R)^c \subset B(w, \delta)^c \cap B(x, \varepsilon)^c$ so that

$$|T_{\mu, \varepsilon}(1_Q b_1, 1_Q b_2)(x) - T_{\mu, \delta}(1_{(2R)^c} 1_Q b_1, 1_Q b_2)(w)|$$

can be dominated by the sum of

$$\int_{(2R)^c} \int_{\mathbb{R}^n} |K(x, y, z) - K(w, y, z)| d\mu(z) d\mu(y) \lesssim \ell(R)^\alpha \int_{R^c} \frac{d\mu(y)}{|x - y|^{m+\alpha}} \lesssim 1$$

and

$$\iint_{\substack{\max(|x-y|, |x-z|) > \varepsilon \\ y \in 2R}} |K(x, y, z)| d\mu(z) d\mu(y) \lesssim \int_{2R} \int \frac{d\mu(z) d\mu(y)}{(\varepsilon + |x - z|)^{2m}} \lesssim \frac{\mu(2R)}{\varepsilon^m} \lesssim 1.$$

Therefore, we have

$$|T_{\mu, \varepsilon}(1_Q b_1, 1_Q b_2)(x)| \lesssim 1 + |T_{\mu, \delta}(1_Q b_1, 1_Q b_2)(w)| + |T_{\mu, \delta}(1_{2R} 1_Q b_1, 1_Q b_2)(w)|.$$

It follows from this by raising to the power $s/4$, averaging over $w \in R$ and raising to power $4/s$ that

$$\begin{aligned} |T_{\mu, \varepsilon}(1_Q b_1, 1_Q b_2)(x)| &\lesssim 1 + M_{\mu, s/4}^Q(1_Q T_{\mu, \delta}(1_Q b_1, 1_Q b_2))(x) \\ &\quad + \left(\frac{1}{\mu(R)} \int_R |T_{\mu, \delta}(1_{2R} 1_Q b_1, 1_Q b_2)(w)|^{s/4} d\mu(w) \right)^{4/s}. \end{aligned}$$

To get the maximal function bound, we also used that R is a cube centred at x and $R \subset Q$.

We have that

$$\int_R |T_{\mu, \delta}(1_{2R \setminus R} 1_Q b_1, 1_Q b_2)(w)|^{s/4} d\mu(w) \lesssim \int_R \left[\int_{2R \setminus R} \frac{d\mu(y)}{|w - y|^m} \right]^{s/4} d\mu(w) \lesssim \mu(R)$$

using that R has small boundary and is doubling (see Lemma 9.44 in [24]).

Next, we bound

$$\begin{aligned} \int_R |T_{\mu,\delta}(1_R b_1, 1_Q b_2)(w)|^{s/4} d\mu(w) &\lesssim \int_R |T_{\mu,\delta}(1_R b_1, 1_R b_2)(w)|^{s/4} d\mu(w) \\ &\quad + \int_R |T_{\mu,\delta}(1_R b_1, 1_{2R \setminus R} 1_Q b_2)(w)|^{s/4} d\mu(w) \\ &\quad + \int_R |T_{\mu,\delta}(1_R b_1, 1_{(2R)^c} 1_Q b_2)(w)|^{s/4} d\mu(w). \end{aligned}$$

Notice that

$$\int_R |T_{\mu,\delta}(1_R b_1, 1_{2R \setminus R} 1_Q b_2)(w)|^{s/4} d\mu(w) \lesssim \int_R \left[\int_{2R \setminus R} \frac{d\mu(z)}{|w-z|^m} \right]^{s/4} d\mu(w) \lesssim \mu(R)$$

and

$$\begin{aligned} \int_R |T_{\mu,\delta}(1_R b_1, 1_{(2R)^c} 1_Q b_2)(w)|^{s/4} d\mu(w) &\lesssim \int_R \left[\int_R \int_{(2R)^c} \frac{d\mu(z)}{|z-w|^{2m}} d\mu(y) \right]^{s/4} d\mu(w) \\ &\lesssim \mu(R) \left[\mu(R) \int_{R^c} \frac{d\mu(z)}{|z-x|^{2m}} \right]^{s/4} \\ &\lesssim \mu(R) \left[\frac{\mu(R)}{\ell(R)^m} \right]^{s/4} \lesssim \mu(R). \end{aligned}$$

It only remains to show that $I \lesssim \mu(R)$ for the term

$$I := \int_R |T_{\mu,\delta}(1_R b_1, 1_R b_2)(w)|^{s/4} d\mu(w).$$

The point simply is that weak type testing implies strong type testing for strictly smaller exponents. Indeed, using (2.2) we see that

$$\begin{aligned} I &= \frac{s}{4} \int_0^\infty \lambda^{s/4-1} \mu(\{w \in R: |T_{\mu,\delta}(1_R b_1, 1_R b_2)(w)| > \lambda\}) d\lambda \\ &\leq \frac{s}{4} \left[\int_0^1 \lambda^{s/4-1} d\lambda + C_0 \int_1^\infty \lambda^{-3s/4-1} d\lambda \right] \mu(R) \lesssim \mu(R). \end{aligned}$$

The desired bound

$$|T_{\mu,\varepsilon}(1_Q b_1, 1_Q b_2)(x)| \lesssim C(\tau) + M_{\mu,s/4}(1_Q T_{\mu,\delta}(1_Q b_1, 1_Q b_2))(x)$$

has now been proved. \square

The following corollary contains the improved testing i.e. testing for $T_{\mu,\sharp}$.

2.3. Corollary. *Let μ be a measure of degree m on \mathbb{R}^n , T be a bilinear m -dimensional SIO and $s > 0$. Let b and t be large enough constants depending only on the dimension n . Suppose $b_i \in L^\infty(\mu)$, $i = 1, 2$, satisfy for every $(2, b)$ -doubling cube R with t -small boundary that*

$$\sup_{\delta > 0} \sup_{\lambda > 0} \lambda^s \mu(\{x \in R: |T_{\mu,\delta}(1_R b_1, 1_R b_2)(x)| > \lambda\}) \lesssim \mu(R).$$

Let $\eta > 0$. For every $(2, b)$ -doubling cube Q with t -small boundary we have that

$$\int_{Q \setminus H_Q(\eta)} [T_{\mu, \#}(1_Q b_1, 1_Q b_2)]^{s/2} d\mu \lesssim C(\eta) \mu(Q)$$

for some boundary region $H_Q(\eta) = Q \setminus (1 - \tau_\eta)Q$ satisfying $\mu(H_Q(\eta)) \leq \eta \mu(Q)$.

Proof. Fix $\eta > 0$ and an arbitrary $(2, b)$ -doubling cube Q with t -small boundary. Let $H_Q(\eta) = Q \setminus (1 - \tau_\eta)Q$, where $\tau_\eta > 0$ is chosen small enough depending on η so that $\mu(H_Q) \leq \eta \mu(Q)$. This is possible, since Q is doubling and of small boundary.

Fix $\delta > 0$. We have by Proposition 2.1 that

$$T_{\mu, \#, \delta}(1_Q b_1, 1_Q b_2)(x) \lesssim C(\eta) + M_{\mu, s/4}^{\mathcal{Q}}(1_Q T_{\mu, \delta}(1_Q b_1, 1_Q b_2))(x)$$

for every $x \in Q \setminus H_Q(\eta)$. Therefore, we have

$$\begin{aligned} \int_{Q \setminus H_Q(\eta)} [T_{\mu, \#, \delta}(1_Q b_1, 1_Q b_2)]^{s/2} d\mu &\lesssim C(\eta) \mu(Q) + \int_{\mathbb{R}^n} M_{\mu}^{\mathcal{Q}}(1_Q |T_{\mu, \delta}(1_Q b_1, 1_Q b_2)|^{s/4})^2 d\mu \\ &\lesssim C(\eta) \mu(Q) + \int_Q |T_{\mu, \delta}(1_Q b_1, 1_Q b_2)|^{s/2} d\mu \\ &\lesssim C(\eta) \mu(Q). \end{aligned}$$

The last estimate used the calculation in Proposition 2.1 showing that weak type testing implies strong type testing for strictly smaller exponents. Letting $\delta \rightarrow 0$ yields by monotone convergence that

$$\int_{Q \setminus H_Q(\eta)} [T_{\mu, \#}(1_Q b_1, 1_Q b_2)]^{s/2} d\mu \lesssim C(\eta) \mu(Q),$$

so we are done. \square

3. SUPPRESSED BILINEAR SINGULAR INTEGRALS

Given a 1-Lipschitz function $\Phi: \mathbb{R}^n \rightarrow [0, \infty)$ we define

$$A_{\Phi}(x, y, z) = \frac{(|x - y| + |x - z|)^{3\beta}}{(|x - y| + |x - z|)^{3\beta} + \Phi(x)^{\beta} \Phi(y)^{\beta} \Phi(z)^{\beta}},$$

where $\beta = \beta(m) = \max(1, 2m/3)$. Given a standard m -dimensional bilinear Calderón–Zygmund kernel K we define the suppressed kernel

$$K_{\Phi}(x, y, z) = A_{\Phi}(x, y, z) K(x, y, z).$$

It is important to understand that $A_{\Phi}(x, y, z) = 1$ if $\Phi(x) = 0$, say.

3.1. Lemma. *The function K_{Φ} is a standard m -dimensional bilinear Calderón–Zygmund kernel with constants independent of the choice of the 1-Lipschitz function Φ . Moreover, K_{Φ} satisfies the following improved size condition*

$$(3.2) \quad |K_{\Phi}(x, y, z)| \lesssim \frac{1}{(|x - y| + |x - z| + \Phi(x) + \Phi(y) + \Phi(z))^{2m}}.$$

Proof. We begin with the size condition (3.2). Let $x, y, z \in \mathbb{R}^n$. We show that

$$(3.3) \quad \begin{aligned} & (|x - y| + |x - z|)^{3\beta} + \Phi(x)^\beta \Phi(y)^\beta \Phi(z)^\beta \\ & \gtrsim (|x - y| + |x - z|)^{3\beta} + \Phi(x)^{3\beta} + \Phi(y)^{3\beta} + \Phi(z)^{3\beta}, \end{aligned}$$

which clearly holds if

$$\max(|x - y|, |x - z|, |y - z|) \geq \frac{1}{2} \max(\Phi(x), \Phi(y), \Phi(z)).$$

Suppose for example that

$$\max(|x - y|, |x - z|, |y - z|) \leq \frac{1}{2} \max(\Phi(x), \Phi(y), \Phi(z)) = \frac{1}{2} \Phi(y).$$

In this case $\Phi(x) \geq \Phi(y) - |x - y| \geq \Phi(y)/2$, and similarly $\Phi(z) \geq \Phi(y)/2$. Hence $\Phi(x)^\beta \Phi(y)^\beta \Phi(z)^\beta \gtrsim \Phi(y)^{3\beta}$, whence

$$\begin{aligned} & (|x - y| + |x - z|)^{3\beta} + \Phi(x)^\beta \Phi(y)^\beta \Phi(z)^\beta \\ & \gtrsim (|x - y| + |x - z|)^{3\beta} + \Phi(x)^{3\beta} + \Phi(y)^{3\beta} + \Phi(z)^{3\beta}. \end{aligned}$$

Using (3.3) we have

$$\begin{aligned} |K_\Phi(x, y, z)| & \lesssim \frac{(|x - y| + |x - z|)^{3\beta - 2m}}{(|x - y| + |x - z|)^{3\beta} + \Phi(x)^{3\beta} + \Phi(y)^{3\beta} + \Phi(z)^{3\beta}} \\ & \lesssim \frac{1}{(|x - y| + |x - z| + \Phi(x) + \Phi(y) + \Phi(z))^{2m}}, \end{aligned}$$

where we applied the fact $\beta \geq 2m/3$.

We turn to the Hölder conditions. We have

$$\begin{aligned} |K_\Phi(x', y, z) - K_\Phi(x, y, z)| & \leq |K(x', y, z)(A_\Phi(x', y, z) - A_\Phi(x, y, z))| \\ & \quad + |(K(x', y, z) - K(x, y, z))A_\Phi(x, y, z)| \\ & =: I + II. \end{aligned}$$

If $|x - x'| \leq \max(|x - y|, |x - z|)/2$, we may use the x -Hölder condition of K to get

$$II \lesssim \frac{|x - x'|^\alpha}{(|x - y| + |x - z|)^{2m + \alpha}},$$

since $|A_\Phi| \leq 1$.

Consider then I . The size estimate of K gives

$$(3.4) \quad I \lesssim \frac{1}{(|x - y| + |x - z|)^{2m}} |\nabla_x A_\Phi(x'', y, z)| |x' - x|,$$

where x'' is a point on the line segment between x and x' .

Define

$$a(x, y, z) := (|x - y| + |x - z|)^{3\beta}$$

and

$$b(x, y, z) := \Phi(x)^\beta \Phi(y)^\beta \Phi(z)^\beta,$$

whence

$$A_\Phi = \frac{a}{a+b}.$$

The gradient $\nabla_x A_\Phi$ can be written as

$$\nabla_x A_\Phi = \frac{(\nabla_x a)(a+b) - a\nabla_x(a+b)}{(a+b)^2} = \frac{(\nabla_x a)b - a\nabla_x b}{(a+b)^2}.$$

Computation of the derivatives gives

$$\nabla_x a(x, y, z) = 3\beta(|x-y| + |x-z|)^{3\beta-1} \left(\frac{x-y}{|x-y|} + \frac{x-z}{|x-z|} \right)$$

and

$$\nabla_x b(x, y, z) = \beta\Phi(x)^{\beta-1}\Phi(y)^\beta\Phi(z)^\beta\nabla_x\Phi(x).$$

Notice that $|\nabla_x\Phi(x)| \leq 1$ since Φ is 1-Lipschitz. These give us the estimates

$$\begin{aligned} |(\nabla_x a)b| &\lesssim (|x-y| + |x-z|)^{3\beta-1}\Phi(x)^\beta\Phi(y)^\beta\Phi(z)^\beta \\ &\leq (|x-y| + |x-z| + \Phi(x) + \Phi(y) + \Phi(z))^{6\beta-1} \end{aligned}$$

and

$$\begin{aligned} |a\nabla_x b| &\lesssim (|x-y| + |x-z|)^{3\beta}\Phi(x)^{\beta-1}\Phi(y)^\beta\Phi(z)^\beta \\ &\leq (|x-y| + |x-z| + \Phi(x) + \Phi(y) + \Phi(z))^{6\beta-1}, \end{aligned}$$

where we used the fact that $\beta - 1 \geq 0$.

Combining the above estimates and using (3.3), we have shown that

$$\begin{aligned} (3.5) \quad |\nabla_x A_\Phi(x, y, z)| &\lesssim \frac{(|x-y| + |x-z| + \Phi(x) + \Phi(y) + \Phi(z))^{6\beta-1}}{((|x-y| + |x-z|)^{3\beta} + \Phi(x)^{3\beta} + \Phi(y)^{3\beta} + \Phi(z)^{3\beta})^2} \\ &\sim \frac{1}{|x-y| + |x-z| + \Phi(x) + \Phi(y) + \Phi(z)}. \end{aligned}$$

Applying this in (3.4) leads to

$$\begin{aligned} I &\lesssim \frac{1}{(|x-y| + |x-z|)^{2m}} \frac{|x' - x|}{|x'' - y| + |x-z| + \Phi(x'') + \Phi(y) + \Phi(z)} \\ &\lesssim \frac{|x' - x|}{(|x-y| + |x-z|)^{2m+1}} \leq \frac{|x' - x|^\alpha}{(|x-y| + |x-z|)^{2m+\alpha}}. \end{aligned}$$

Hence K_Φ satisfies the x -Hölder estimate. In the same way one shows that K_Φ satisfies the other Hölder estimates. \square

We define in the natural way

$$\begin{aligned} T_{\mu, \Phi, \varepsilon}(f, g)(x) &= \iint_{\max(|x-y|, |x-z|) > \varepsilon} K_{\Phi}(x, y, z) f(y) g(z) d\mu(y) d\mu(z); \\ T_{\mu, \Phi, \# , \delta}(f, g)(x) &= \sup_{\varepsilon > \delta} |T_{\mu, \Phi, \varepsilon}(f, g)(x)|; \\ T_{\mu, \Phi, \#}(f, g)(x) &= T_{\mu, \Phi, \# , 0}(f, g)(x). \end{aligned}$$

The following proposition is one of the key reasons why the suppressed operators are useful.

3.6. Proposition. *Let μ be a measure of degree m on \mathbb{R}^n and T be a bilinear m -dimensional SIO. For a given 1-Lipschitz function Φ there holds that*

$$(3.7) \quad T_{\mu, \Phi, \#}(f, g)(x) \leq T_{\mu, \Phi, \Phi(x)}(f, g)(x) + CM_{\mu}f(x)M_{\mu}g(x).$$

Proof. Fix $\delta > 0$ for which we will control $|T_{\mu, \Phi, \delta}(f, g)(x)|$ with a bound independent of δ . Assume first that $\Phi(x) \geq \delta$. Then we have that

$$\begin{aligned} T_{\mu, \Phi, \delta}(f, g)(x) &= \iint_{\max(|x-y|, |x-z|) > 2\Phi(x)} K_{\Phi}(x, y, z) f(y) g(z) d\mu(y) d\mu(z) \\ &\quad + \iint_{\delta < \max(|x-y|, |x-z|) \leq 2\Phi(x)} K_{\Phi}(x, y, z) f(y) g(z) d\mu(y) d\mu(z). \end{aligned}$$

Notice that

$$\begin{aligned} &\iint_{\max(|x-y|, |x-z|) \leq 2\Phi(x)} |K_{\Phi}(x, y, z) f(y) g(z)| d\mu(y) d\mu(z) \\ &\lesssim \int_{\bar{B}(x, 2\Phi(x))} \int_{\mathbb{R}^n} \frac{|f(y)g(z)|}{(\Phi(x) + |x-z|)^{2m}} d\mu(z) d\mu(y) \\ &\lesssim M_{\mu}g(x) \cdot \frac{1}{\Phi(x)^m} \int_{B(x, 3\Phi(x))} |f(y)| d\mu(y) \lesssim M_{\mu}f(x)M_{\mu}g(x). \end{aligned}$$

We also bound

$$\begin{aligned} &\left| \iint_{\max(|x-y|, |x-z|) > 2\Phi(x)} K_{\Phi}(x, y, z) f(y) g(z) d\mu(y) d\mu(z) \right| \\ &\leq \sup_{\varepsilon > \Phi(x)} \left| \iint_{\max(|x-y|, |x-z|) > \varepsilon} K_{\Phi}(x, y, z) f(y) g(z) d\mu(y) d\mu(z) \right| = T_{\mu, \Phi, \# , \Phi(x)}(f, g)(x). \end{aligned}$$

If it happens that $\Phi(x) < \delta$ we obviously have the bound

$$|T_{\mu, \Phi, \delta}(f, g)(x)| \leq T_{\mu, \Phi, \Phi(x)}(f, g)(x).$$

So we have shown that for every $x \in \mathbb{R}^n$ there holds that

$$|T_{\mu, \Phi, \delta}(f, g)(x)| \leq T_{\mu, \Phi, \Phi(x)}(f, g)(x) + CM_{\mu}f(x)M_{\mu}g(x).$$

Therefore, we are done if we show that

$$(3.8) \quad T_{\mu, \Phi, \# , \Phi(x)}(f, g)(x) \leq T_{\mu, \Phi, \Phi(x)}(f, g)(x) + CM_{\mu}f(x)M_{\mu}g(x).$$

To this end, we fix $\varepsilon > \Phi(x)$ and shall control $|T_{\mu, \Phi, \varepsilon}(f, g)(x)|$ with a bound independent of ε . Since now

$$\left| \iint_{\max(|x-y|, |x-z|) > \varepsilon} K(x, y, z) f(y) g(z) d\mu(y) d\mu(z) \right| \leq T_{\mu, \Phi, \varepsilon}(f, g)(x),$$

the equation (3.8) follows from showing that

$$\iint_{\max(|x-y|, |x-z|) > \varepsilon} |K(x, y, z) - K_{\Phi}(x, y, z)| |f(y)| |g(z)| d\mu(y) d\mu(z) \lesssim M_{\mu} f(x) M_{\mu} g(x).$$

Notice that

$$1 - A_{\Phi}(x, y, z) \leq \frac{\Phi(x)^{\beta} \Phi(y)^{\beta} \Phi(z)^{\beta}}{(|x-y| + |x-z|)^{3\beta}} \lesssim \sum_{j=1}^3 \frac{\varepsilon^{j\beta}}{(|x-y| + |x-z|)^{j\beta}}$$

by using the definition of $A_{\Phi}(x, y, z)$, the 1-Lipschitz property of Φ and the fact that $\Phi(x) < \varepsilon$. This implies that

$$\begin{aligned} & \iint_{\max(|x-y|, |x-z|) > \varepsilon} |K(x, y, z) - K_{\Phi}(x, y, z)| |f(y)| |g(z)| d\mu(y) d\mu(z) \\ & \lesssim \sum_{j=1}^3 \varepsilon^{j\beta} \iint_{\max(|x-y|, |x-z|) > \varepsilon} \frac{|f(y)| |g(z)|}{(|x-y| + |x-z|)^{2m+j\beta}} d\mu(y) d\mu(z) \lesssim M_{\mu} f(x) M_{\mu} g(x), \end{aligned}$$

completing the proof. \square

3.9. Remark. It follows that for every 1-Lipschitz function the operator $T_{\mu, \Phi, \#}$ is bounded $L^p(\mu) \times L^q(\mu) \rightarrow L^r(\mu)$, if $T_{\mu, \#}$ is. In particular, the weak boundedness property involving $T_{\mu, \Phi, \delta}$ is a reasonable condition.

3.1. L^{∞} suppression. We now indicate how the key estimate (3.7) allows us – in a proper sense – to extend L^{∞} properties from a given set to the whole space. In this section let μ be a *finite* measure of order m .

Suppose f_0, g_0 are some fixed functions satisfying $|f_0|, |g_0| \leq 1$. Notice that $T_{\mu, \varepsilon}(f_0, g_0)(x)$ is, for every $x \in \mathbb{R}^n$ and $\varepsilon > 0$, well-defined as an absolutely convergent integral (since μ is finite). Let S_0 consist of those $x \in \mathbb{R}^n$ for which it holds that

$$T_{\mu, \#}(f_0, g_0)(x) > \lambda_0.$$

Here $\lambda_0 > 0$ is some fixed constant. This means that $1_{\mathbb{R}^n \setminus S_0} T_{\mu, \#}(f_0, g_0) \leq \lambda_0$ – a property which we would like to have also in S_0 . Of course, just the opposite holds in S_0 ! However, if we choose Φ appropriately, then for some absolute constant C we have $T_{\mu, \Phi, \#}(f_0, g_0) \leq \lambda_0 + C$ everywhere. Let us see how come. Notice that in $\mathbb{R}^n \setminus S_0$ everything is fine with any choice of Φ . Indeed, if $x \in \mathbb{R}^n \setminus S_0$, then simply

$$T_{\mu, \Phi, \#}(f_0, g_0)(x) \leq T_{\mu, \#}(f_0, g_0)(x) + C \leq \lambda_0 + C.$$

When controlling what happens in S_0 the choice of Φ becomes very relevant.

Define

$$\varepsilon(x) = \sup\{\varepsilon > 0: |T_{\mu,\varepsilon}(f_0, g_0)(x)| > \lambda_0\}.$$

If $x \in S_0$ then obviously $\varepsilon(x) > 0$. It also holds that $\varepsilon(x) < \infty$, since we have $\lim_{\varepsilon \rightarrow \infty} T_{\mu,\varepsilon}(f_0, g_0)(x) = 0$ by monotone convergence. Define

$$S = \bigcup_{x \in S_0} B(x, \varepsilon(x)).$$

If Φ is any 1-Lipschitz function satisfying that $\Phi(x) \geq d(x, S^c)$, then we are in business. To see this simply note that if $x \in S_0$, then $\Phi(x) \geq \varepsilon(x)$, and so

$$T_{\mu,\Phi,\#}(f_0, g_0)(x) \leq T_{\mu,\Phi,\#}(f_0, g_0)(x) + C \leq \lambda_0 + C.$$

We have shown that $T_{\mu,\Phi,\#}(f_0, g_0) \leq \lambda_0 + C$ everywhere.

This is certainly extremely convenient. Of course, only if S is not some horribly large set! To prevent this from happening we need some additional, but rather weak, assumptions. It is enough that for some $s > 0$ and $C_0 < \infty$ we have

$$\sup_{\lambda > 0} \lambda^s \mu(\{x \in \mathbb{R}^n \setminus H: T_{\mu,\#}(f_0, g_0)(x) > \lambda\}) \leq C_0 \mu(\mathbb{R}^n)$$

for some set H satisfying that $\mu(H) \leq \eta_0 \mu(\mathbb{R}^n)$, $\eta_0 < 1$. We also need to choose $\lambda_0 \lesssim 1$ large enough. Indeed, we will show that for all large enough λ_0 we have

$$(3.10) \quad S \subset \{x \in \mathbb{R}^n: T_{\mu,\#}(f_0, g_0)(x) > \lambda_0/2\},$$

and so

$$\mu(S \setminus H) \leq \frac{2^s C_0}{\lambda_0^s} \mu(\mathbb{R}^n).$$

This allows us to make sure that for large enough λ_0 we have $\mu(S^c) \sim \mu(\mathbb{R}^n)$, since $\mu(H) \leq \eta_0 \mu(\mathbb{R}^n)$.

Let us show (3.10). Let $x \in S$. Then there exists a point $x_0 \in S_0$ and a radius ε_0 such that $x \in B(x_0, \varepsilon_0)$ and $|T_{\mu,\varepsilon_0}(f_0, g_0)(x_0)| > \lambda_0$. The claim follows once we show that $|T_{\mu,\varepsilon_0}(f_0, g_0)(x) - T_{\mu,\varepsilon_0}(f_0, g_0)(x_0)| \lesssim 1$. We have

$$\begin{aligned} & |T_{\mu,\varepsilon_0}(f_0, g_0)(x) - T_{\mu,\varepsilon_0}(f_0, g_0)(x_0)| \\ & \leq |T_{\mu,\varepsilon_0}(1_{B(x_0, 2\varepsilon_0)} f_0, g_0)(x)| + |T_{\mu,\varepsilon_0}(1_{B(x_0, 2\varepsilon_0)} f_0, g_0)(x_0)| \\ & \quad + |T_{\mu,\varepsilon_0}(1_{B(x_0, 2\varepsilon_0)^c} f_0, g_0)(x) - T_{\mu,\varepsilon_0}(1_{B(x_0, 2\varepsilon_0)^c} f_0, g_0)(x_0)|. \end{aligned}$$

Applying the size condition of the kernel there holds

$$\begin{aligned} |T_{\mu,\varepsilon_0}(1_{B(x_0, 2\varepsilon_0)} f_0, g_0)(x)| & \lesssim \iint \frac{1_{B(x_0, 2\varepsilon_0)}(y)}{(\varepsilon_0 + |x - z|)^{2m}} d\mu(y) d\mu(z) \\ & \lesssim \frac{\mu(B(x_0, 2\varepsilon_0))}{\varepsilon_0^m} \lesssim 1. \end{aligned}$$

The corresponding term evaluated at x_0 is estimated in the same way. The difference can be estimated as follows

$$\begin{aligned} & |T_{\mu, \varepsilon_0}(1_{B(x_0, 2\varepsilon_0)^c} f_0, g_0)(x) - T_{\mu, \varepsilon_0}(1_{B(x_0, 2\varepsilon_0)^c} f_0, g_0)(x_0)| \\ &= \left| \iint (K(x, y, z) - K(x_0, y, z)) 1_{B(x_0, 2\varepsilon_0)^c}(y) f_0(y) g_0(z) d\mu(y) d\mu(z) \right| \\ &\lesssim \iint \frac{\varepsilon_0^\alpha 1_{B(x_0, 2\varepsilon_0)^c}(y)}{(|x_0 - y| + |x_0 - z|)^{2m+\alpha}} d\mu(y) d\mu(z) \lesssim 1. \end{aligned}$$

We have shown (3.10).

In practice, we will need to do the above with slightly more generality. We formulate this as a separate proposition.

3.11. Proposition. *Suppose μ is a finite measure of order m and T^1, \dots, T^k , $k \lesssim 1$, are bilinear m -dimensional SIO. Let the functions $(f_0^1, g_0^1), \dots, (f_0^k, g_0^k)$ satisfy $|f_0^i|, |g_0^i| \lesssim 1$ for all $i = 1, \dots, k$. Suppose that for some $s > 0$, $C_0 < \infty$ and for some set H satisfying that $\mu(H) \leq \eta_0 \mu(\mathbb{R}^n)$, $\eta_0 < 1$, we have for every $i = 1, \dots, k$ that*

$$\sup_{\lambda > 0} \lambda^s \mu(\{x \in \mathbb{R}^n \setminus H : T_{\mu, \#}^i(f_0^i, g_0^i)(x) > \lambda\}) \leq C_0 \mu(\mathbb{R}^n).$$

Then there exists a 1-Lipschitz function Φ_0 with

$$\mu(\{\Phi_0 = 0\}) \sim \mu(\mathbb{R}^n),$$

so that for all 1-Lipschitz functions $\Phi \geq \Phi_0$ there holds

$$T_{\mu, \Phi, \#}^i(f_0^i, g_0^i)(x) \lesssim 1, \quad x \in \mathbb{R}^n,$$

for every $i = 1, \dots, k$.

Proof. Let S_i be the suppression sets with parameter λ_0 when we apply the above suppression procedure with T replaced by T^i and f_0, g_0 replaced with f_0^i, g_0^i . Define

$$\Phi_0(x) = d(x, (S_1 \cup \dots \cup S_k)^c),$$

and let Φ be a 1-Lipschitz function such that $\Phi \geq \Phi_0$. In particular, $\Phi(x) \geq d(x, S_i^c)$ and so $T_{\mu, \Phi, \#}^i(f_0^i, g_0^i)(x) \leq \lambda_0 + C$ for every x and $i = 1, \dots, k$. Suppose that $\lambda_0 \lesssim 1$ is fixed large enough. Then

$$\mu(S_i \setminus H) \leq \frac{2^s C_0}{\lambda_0^s} \mu(\mathbb{R}^n) \leq \frac{1 - \eta_0}{2k} \mu(\mathbb{R}^n)$$

and we get

$$\mu((S_1 \cup \dots \cup S_k)^c) \geq \frac{1 - \eta_0}{2} \mu(\mathbb{R}^n).$$

□

3.12. Remark. Given a bilinear m -dimensional SIO T this proposition will be later applied with $k = 3$, $T^1 = T$, $T^2 = T^{1*}$, $T^3 = T^{2*}$, and to some accretive L^∞ functions.

This completes our explanation of the L^∞ suppression techniques. These ideas and the above calculations will be used concretely to prove a certain bilinear big pieces Tb theorem, which is the key to proving our main Tb theorem.

4. THE BIG PIECE Tb

In this section we prove Theorem 4.2 – a very particular big piece type Tb theorem adapted to our needs.

4.1. Remark. In previous literature many summation arguments in Tb theorems were based on summing the numbers

$$\delta(J, R) := \frac{\ell(J)^{\alpha/2} \ell(R)^{\alpha/2}}{D(J, R)^{m+\alpha}}, \quad D(J, R) := \ell(J) + \ell(R) + d(J, R),$$

over all dyadic cubes. This was done in the ℓ^2 sense by Nazarov–Treil–Volberg. We can also prove the L^p analog, which goes as follows. Let μ be a measure of order m on \mathbb{R}^n , and $\mathcal{D}, \mathcal{D}'$ be two dyadic grids on \mathbb{R}^n . For every $s \in (1, \infty)$ and $x_J \geq 0, J \in \mathcal{D}$, we have

$$\left\| \left(\sum_{R \in \mathcal{D}'} 1_R \left[\sum_{J \in \mathcal{D}} \delta(J, R) \mu(J) x_J \right]^2 \right)^{1/2} \right\|_{L^s(\mu)} \lesssim \left\| \left(\sum_{J \in \mathcal{D}} x_J^2 1_J \right)^{1/2} \right\|_{L^s(\mu)}.$$

However, we noticed a new simpler way to sum all the relevant parts in the Tb argument, and no longer rely on this result.

4.2. Theorem. Let μ be a measure of order m such that $\mu(\mathbb{R}^n \setminus Q_0) = 0$ for some cube $Q_0 \subset \mathbb{R}^n$. Assume also that for some $t_0 < \infty$ we have for every $\lambda > 0$ that

$$\mu(\{x \in Q_0 : d(x, \partial Q_0) \leq \lambda \ell(Q_0)\}) \leq t_0 \lambda \mu(Q_0).$$

Let T be a bilinear m -dimensional SIO, and let $b_i \in L^\infty(\mu), i = 1, 2, 3$, be such that

$$|\langle b_i \rangle_Q^\mu| \gtrsim 1 \quad \text{for all cubes } Q \subset Q_0.$$

We assume the weak boundedness property in the form that

$$|\langle T_{\mu, \Phi, \delta}(1_Q b_1, 1_Q b_2), 1_Q b_3 \rangle_\mu| \lesssim \mu(5Q) \quad \text{for all cubes } Q \text{ satisfying } 5Q \subset Q_0$$

uniformly over the choice of the 1-Lipschitz function $\Phi: \mathbb{R}^n \rightarrow [0, \infty)$ and the truncation parameter $\delta > 0$. Let $s > 0$. Assume that there is a set $H \subset \mathbb{R}^n$ so that $\mu(H) \leq \eta_0 \mu(Q_0)$ for some $\eta_0 < 1$ and so that the following three testing conditions hold:

$$\sup_{\lambda > 0} \lambda^s \mu(\{x \in Q_0 \setminus H : S_{\mu, \#}(b, b')(x) > \lambda\}) \lesssim \mu(Q_0)$$

for all the choices $(S, b, b') \in \{(T, b_1, b_2), (T^{1*}, b_3, b_2), (T^{2*}, b_1, b_3)\}$.

Then there is a set $G \subset Q_0$ so that $\mu(G) \sim \mu(Q_0)$ and the following holds. For every $1 < p, q, r < \infty$ satisfying $1/p + 1/q = 1/r$ we have uniformly for functions $f \in L^p(\mu), g \in L^q(\mu)$ and $h \in L^{r'}(\mu)$ supported in G that

$$(4.3) \quad \sup_{\varepsilon > 0} |\langle T_{\mu, \varepsilon}(f, g), h \rangle_\mu| \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)}.$$

Proof. We begin by reducing the desired estimate to the boundedness of a certain suppressed operator. Let us apply Proposition 3.11 with $T_{\mu,\#}(b_1, b_2)$, $T_{\mu,\#}^{1*}(b_3, b_2)$ and $T_{\mu,\#}^{2*}(b_1, b_3)$, and let Φ_0 be the resulting 1-Lipschitz function. For reasons that will become clear later, we have to modify the function Φ_0 a little. Fix a small number $\lambda_0 > 0$ so that

$$t_0 \lambda_0 \mu(Q_0) \leq \frac{\mu(\{\Phi_0 = 0\})}{2}.$$

Clearly, we can choose λ_0 so that it only depends on the constants in our assumptions, since $\mu(\{\Phi_0 = 0\}) \sim \mu(Q_0)$. The choice of λ_0 implies by the small boundary assumption of Q_0 that

$$(4.4) \quad \mu(\{x \in Q_0 : d(x, \partial Q_0) \leq \lambda_0 \ell(Q_0)\}) \leq \frac{\mu(\{\Phi_0 = 0\})}{2}.$$

Let ϕ be the 1-Lipschitz function

$$\phi(x) := \max(\lambda_0 \ell(Q_0) - d(x, \partial Q_0), 0).$$

We have by (4.4) that

$$\mu(\{\phi \neq 0\}) \leq \frac{\mu(\{\Phi_0 = 0\})}{2}.$$

Moreover,

$$\phi(x) \geq \frac{\lambda_0 \ell(Q_0)}{2} \quad \text{if } d(x, \partial Q_0) \leq \lambda_0 \ell(Q_0)/2.$$

Define $\Phi_1 := \max(\Phi_0, \phi)$. Then Φ_1 is a 1-Lipschitz function, and the properties that we just verified for ϕ give that

$$\mu(\{\Phi_1 = 0\}) \geq \frac{\mu(\{\Phi_0 = 0\})}{2}$$

and

$$(4.5) \quad \Phi_1(x) \geq \frac{\lambda_0 \ell(Q_0)}{2} \quad \text{if } d(x, \partial Q_0) \leq \frac{\lambda_0 \ell(Q_0)}{2}.$$

We define the set G by setting

$$G = \{x \in Q_0 : \Phi_1(x) = 0\}.$$

Since $\mu(G) \sim \mu(Q_0)$ it only remains to check (4.3).

To this end, fix an arbitrary truncation parameter $\varepsilon > 0$ and exponents $p, q, r \in (1, \infty)$ so that $1/p + 1/q = 1/r$. The 1-Lipschitz function that we will use in the suppression is defined by

$$\Phi = \max(\varepsilon, \Phi_1).$$

Since $\Phi \geq \Phi_0$, Proposition 3.11 shows that

$$T_{\mu,\Phi,\#}(b_1, b_2) + T_{\mu,\Phi,\#}^{1*}(b_3, b_2) + T_{\mu,\Phi,\#}^{2*}(b_1, b_3) \lesssim 1.$$

We write $T_{\mu,\Phi} = T_{\mu,\Phi,0}$ which makes sense because $\Phi(x) \geq \varepsilon$ for every x .

Suppose $f \in L^p(\mu)$, $g \in L^q(\mu)$ and let $x \in G$. Since $\Phi(x) = \varepsilon$, we have

$$\begin{aligned} |T_{\mu,\varepsilon}(f, g)(x) - T_{\mu,\Phi}(f, g)(x)| &\leq |T_{\mu,\varepsilon}(f, g)(x) - T_{\mu,\Phi,\Phi(x)}(f, g)(x)| \\ &\quad + |T_{\mu,\Phi,\Phi(x)}(f, g)(x) - T_{\mu,\Phi}(f, g)(x)| \\ &\lesssim M_\mu f(x) M_\mu g(x). \end{aligned}$$

The required estimates for the final step can easily be read from the proof of Proposition 3.11. This shows that if $f \in L^p(\mu)$, $g \in L^q(\mu)$ and $h \in L^{r'}(\mu)$ are functions supported in G , then

$$\begin{aligned} |\langle T_{\mu,\varepsilon}(f, g), h \rangle_\mu| &\leq C \|M_\mu f M_\mu g\|_{L^r(\mu)} \|h\|_{L^{r'}(\mu)} + |\langle T_{\mu,\Phi}(f, g), h \rangle_\mu| \\ &\leq C \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)} + |\langle T_{\mu,\Phi}(f, g), h \rangle_\mu|. \end{aligned}$$

Thus, the required estimate (4.3) follows, if we show that there exists an absolute constant C , depending only on the constants in the assumptions, so that

$$(4.6) \quad |\langle T_{\mu,\Phi}(f, g), h \rangle_\mu| \leq C \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)}$$

whenever $f \in L^p(\mu)$, $g \in L^q(\mu)$ and $h \in L^{r'}(\mu)$ (not necessarily supported in G anymore). We denote the best such constant C by $\|T_{\mu,\Phi}\|$. Notice that we a priori know that $\|T_{\mu,\Phi}\| < \infty$, because $\Phi(x) \geq \varepsilon$ for every $x \in Q_0$. We are after the quantitative bound.

4.7. *Remark.* Since $\Phi_1(x) = 0$ for $x \in G$, we directly have the identity

$$\langle T_{\mu,\varepsilon}(f, g), h \rangle_\mu = \langle T_{\mu,\Phi_1,\varepsilon}(f, g), h \rangle_\mu$$

if the functions are supported in G . However, we reduced to the operator $T_{\mu,\Phi}$ in order to get rid of the truncation present in $T_{\mu,\Phi_1,\varepsilon}$. This is convenient – see the proof of Lemma 4.20 to understand that zero average is easier to utilise if there are no truncations in the integration.

We start aiming towards (4.6). For the moment, we fix three functions $f \in L^p(\mu)$, $g \in L^q(\mu)$ and $h \in L^{r'}(\mu)$ with norm at most 1 so that

$$|\langle T_{\mu,\Phi}(f, g), h \rangle_\mu| \geq \frac{\|T_{\mu,\Phi}\|}{2}.$$

Define

$$Q_{0,\partial} := \{x \in Q_0 : d(x, \partial Q_0) < \lambda_0 \ell(Q_0)/2\}.$$

We may write $f = f_\partial + f_{\text{int}}$, where $f_\partial := 1_{Q_{0,\partial}} f$ and $f_{\text{int}} := f - f_\partial$. The functions $h_\partial, g_\partial, h_{\text{int}}$ and g_{int} are defined similarly giving us the decomposition

$$\begin{aligned} \langle T_{\mu,\Phi}(f, g), h \rangle_\mu &= \langle T_{\mu,\Phi}(f_\partial, g), h \rangle_\mu + \langle T_{\mu,\Phi}(f_{\text{int}}, g_\partial), h \rangle_\mu \\ &\quad + \langle T_{\mu,\Phi}(f_{\text{int}}, g_{\text{int}}), h_\partial \rangle_\mu + \langle T_{\mu,\Phi}(f_{\text{int}}, g_{\text{int}}), h_{\text{int}} \rangle_\mu. \end{aligned}$$

The last term is the main one. The first three terms are handled using the suppression in the boundary region $Q_{0,\partial}$, where at least one of the appearing functions

is supported in. For example, we have by (4.5) and the improved size condition (3.2) that

$$|\langle T_{\mu,\Phi}(f_{\text{int}}, g_{\partial}), h \rangle_{\mu}| \lesssim \frac{\|f_{\text{int}}\|_{L^1(\mu)} \|g_{\partial}\|_{L^1(\mu)} \|h\|_{L^1(\mu)}}{\ell(Q_0)^{2m}} \lesssim 1.$$

The last step follows from Hölder's inequality using the fact that μ is of order m . Thus, we have shown that there exists a constant C , depending only on the constants in our assumptions, so that

$$(4.8) \quad \frac{\|T_{\mu,\Phi}\|}{2} \leq C + \sup_{f,g,h} |\langle T_{\mu,\Phi}(f, g), h \rangle_{\mu}|,$$

where the supremum is now over functions $f \in L^p(\mu)$, $g \in L^q(\mu)$ and $h \in L^{r'}(\mu)$ with norm at most one and *supported in* $Q_0 \setminus Q_{0,\partial}$. The reason why we reduced to functions supported in $Q_0 \setminus Q_{0,\partial}$ is related to the fact that we can only use cubes which are inside Q_0 in our upcoming martingale decompositions. This is dictated by our assumptions.

We shall choose three such functions and split them using b -adapted martingales, and then perform the standard averaging argument of Nazarov–Treil–Volberg [23] to reduce to “good” functions.

Adapted martingales. At this point we need to recall the random dyadic grids (these facts are essentially presented in this way by Hytönen [7]). Let \mathcal{D}_{st} denote the standard dyadic grid, consisting of all the cubes of the form $2^{-k}(\ell + [0, 1)^n)$, where $k \in \mathbb{Z}$ and $\ell \in \mathbb{Z}^n$. A generic dyadic grid, parametrized by

$$\omega \in \Omega := (\{0, 1\}^n)^{\mathbb{Z}},$$

is of the form

$$\mathcal{D}(\omega) = \cup_{k \in \mathbb{Z}} \mathcal{D}_k(\omega), \text{ where } \mathcal{D}_k(\omega) = \{Q + x_k^{\omega} : Q \in \mathcal{D}_{st,k}\} \text{ and } x_k^{\omega} = \sum_{j>k} \omega_j 2^{-j}.$$

We get random dyadic grids by placing the natural product probability measure \mathbb{P}_{ω} on $\Omega = (\{0, 1\}^n)^{\mathbb{Z}}$ (thus the coordinate functions ω_j are independent and $\mathbb{P}_{\omega}(\omega_j = \eta) = 2^{-n}$ if $\eta \in \{0, 1\}^n$).

Let us consider some $\omega \in \Omega$ for the moment. Choose the integer u_0 so that $2^{u_0} < \lambda_0 \ell(Q_0)/4 \leq 2^{u_0+1}$. We define the shorthand

$$\mathcal{D}_0(\omega) := \{Q \in \mathcal{D}(\omega) : Q \subset Q_0, \ell(Q) \leq 2^{u_0}\}.$$

Suppose f is a locally μ -integrable function. Define for every $Q \in \mathcal{D}_0(\omega)$ the numbers

$$E_Q^1 f := \frac{\langle f \rangle_Q^{\mu}}{\langle b_1 \rangle_Q^{\mu}}.$$

If $\ell(Q) = 2^{u_0-1}$, we set

$$D_Q^1 f := E_Q^1 f,$$

and if $\ell(Q) < 2^{u_0-1}$, then

$$D_Q^1 f := E_Q^1 f - E_{Q^{(1)}}^1 f.$$

Using these numbers the b_1 -adapted martingale difference operators Δ_Q^1 are defined for every $Q \in \mathcal{D}_0(\omega)$ by setting

$$\Delta_Q^1 f := \sum_{Q' \in \text{ch}(Q)} (D_{Q'}^1 f) 1_{Q'} b_1.$$

Notice the difference depending on whether $\ell(Q) = 2^{u_0}$ or $\ell(Q) < 2^{u_0}$.

We will also need the adjoint operators of Δ_Q^1 . Namely, for $Q \in \mathcal{D}_0(\omega)$ with $\ell(Q) = 2^{u_0}$ define

$$\Delta_Q^{1*} f := \sum_{Q' \in \text{ch}(Q)} \frac{\langle f b_1 \rangle_{Q'}^\mu}{\langle b_1 \rangle_{Q'}^\mu} 1_{Q'},$$

and for $Q \in \mathcal{D}_0(\omega)$ with $\ell(Q) < 2^{u_0}$ define

$$\Delta_Q^{1*} f := \sum_{Q' \in \text{ch}(Q)} \frac{\langle f b_1 \rangle_{Q'}^\mu}{\langle b_1 \rangle_{Q'}^\mu} 1_{Q'} - \frac{\langle f b_1 \rangle_Q^\mu}{\langle b_1 \rangle_Q^\mu} 1_Q.$$

We list some properties of these martingale differences. Let $Q \in \mathcal{D}_0(\omega)$. If $\ell(Q) < 2^{u_0}$, then $\int \Delta_Q^1 f d\mu = 0$. For two functions f and g we have $\langle \Delta_Q^1 f, g \rangle_\mu = \langle f, \Delta_Q^{1*} g \rangle_\mu$. If $R \in \mathcal{D}_0(\omega)$ is another cube, there holds

$$(4.9) \quad \Delta_Q^1 \Delta_R^1 f = \begin{cases} 0, & \text{if } Q \neq R, \\ \Delta_Q^1 f, & \text{if } Q = R, \end{cases}$$

and

$$\Delta_Q^{1*} \Delta_R^{1*} f = \begin{cases} 0, & \text{if } Q \neq R, \\ \Delta_Q^{1*} f, & \text{if } Q = R. \end{cases}$$

Suppose now $s \in (1, \infty)$. Let $f \in L^s(\mu)$ be a function whose support can be covered with cubes in $\mathcal{D}_0(\omega)$ of side length 2^{u_0} . Then f can be represented as

$$f = \sum_{Q \in \mathcal{D}_0(\omega)} \Delta_Q^1 f = \sum_{Q \in \mathcal{D}_0(\omega)} \Delta_Q^{1*} f,$$

where the convergence takes place unconditionally (that is, independently of the order) in $L^s(\mu)$. Define the b_1 -adapted dyadic square functions

$$S_\omega^1 f := \left(\sum_{\substack{Q \in \mathcal{D}_0(\omega) \\ \ell(Q) < 2^{u_0}}} |D_Q^1 f|^2 1_Q \right)^{1/2} \quad \text{and} \quad S_\omega^{1*} f := \left(\sum_{Q \in \mathcal{D}_0(\omega)} |\Delta_Q^{1*} f|^2 \right)^{1/2}.$$

We have the standard estimates

$$(4.10) \quad \|f\|_{L^s(\mu)} \lesssim \| (S_\omega^1 f) b_1 \|_{L^s(\mu)} \lesssim \| S_\omega^1 f \|_{L^s(\mu)} \lesssim \|f\|_{L^s(\mu)}$$

and

$$(4.11) \quad \|f\|_{L^s(\mu)} \sim \| S_\omega^{1*} f \|_{L^s(\mu)}.$$

In (4.10) the middle step is trivial using $b_1 \in L^\infty(\mu)$; the first and the last inequalities are the main facts. Notice the following consequence of this: If $\{a_Q\}_{Q \in \mathcal{D}_0(\omega)}$ is a collection of real numbers so that

$$\left(\sum_{Q \in \mathcal{D}_0(\omega)} |a_Q \Delta_Q^1 f|^2 \right)^{1/2}$$

is in $L^s(\mu)$, then $g := \sum_{Q \in \mathcal{D}_0(\omega)} a_Q \Delta_Q^1 f$ is well defined in $L^s(\mu)$. For every $Q \in \mathcal{D}_0(\omega)$ there holds by (4.9) that $\Delta_Q^1 g = a_Q \Delta_Q^1 f$, and accordingly

$$(4.12) \quad \left\| \sum_{Q \in \mathcal{D}_0(\omega)} a_Q \Delta_Q^1 f \right\|_{L^s(\mu)} \sim \left\| \left(\sum_{Q \in \mathcal{D}_0(\omega)} |a_Q \Delta_Q^1 f|^2 \right)^{1/2} \right\|_{L^s(\mu)}.$$

The corresponding observation holds also with the operators Δ_Q^{1*} .

For later use, we define the operators

$$\begin{aligned} E_{\omega, 2^k}^1 f &:= \sum_{\substack{Q \in \mathcal{D}_0(\omega) \\ \ell(Q) = 2^k}} (E_Q^1 f) 1_Q b_1, \quad k \leq u_0, \\ D_{\omega, 2^{u_0}}^1 f &:= E_{2^{u_0-1}}^1 f, \\ D_{\omega, 2^k}^1 f &:= E_{\omega, 2^{k-1}}^1 f - E_{\omega, 2^k}^1 f, \quad k < u_0. \end{aligned}$$

The corresponding definitions can be made using the functions b_2 and b_3 too. Then we simply replace the super index 1 above by 2 or 3 depending on the case.

Next, we recall the good and bad cubes of Nazarov-Treil-Volberg [23]. If $\omega \in \Omega$, we say that a cube Q is ω -good with parameters $(\gamma, \sigma) \in (0, 1) \times \mathbb{Z}_+$ if

$$d(Q, R) > \ell(Q)^\gamma \ell(R)^{1-\gamma} \quad \text{for all } R \in \mathcal{D}(\omega) \text{ with } \ell(R) \geq 2^\sigma \ell(Q);$$

otherwise Q is said to be ω -bad. If $\omega_1, \omega_2 \in \Omega$, we say that Q is (ω_1, ω_2) -good if it is *both* ω_1 -good *and* ω_2 -good; otherwise Q is said to be (ω_1, ω_2) -bad. From now on, we fix $\gamma \in (0, 1)$. The parameter σ will be fixed during the probabilistic argument below to be large enough but still $\lesssim 1$.

Let again $s \in (1, \infty)$ and $f \in L^s(\mu)$ with support in $Q_0 \setminus Q_{0, \partial}$. For $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega \times \Omega \times \Omega$ we define

$$P_B^1(\omega) f := \sum_{\substack{Q \in \mathcal{D}_0(\omega_1) \\ Q \text{ is } (\omega_2, \omega_3)\text{-bad}}} \Delta_Q^1 f$$

and

$$P_G^1(\omega) f := \sum_{\substack{Q \in \mathcal{D}_0(\omega_1) \\ Q \text{ is } (\omega_2, \omega_3)\text{-good}}} \Delta_Q^1 f,$$

where we keep in mind the dependence of these definitions on the (not yet fixed) goodness parameter σ . We also define $P_B^2(\omega)$ using the operators Δ_Q^2 with cubes $Q \in \mathcal{D}_0(\omega_2)$ that are (ω_1, ω_3) -bad and $P_G^3(\omega)$ using the operators Δ_Q^3 with cubes $Q \in \mathcal{D}_0(\omega_3)$ that are (ω_1, ω_2) -good and so on.

On average, the norm of the bad part is small. The L^2 case is by Nazarov-Treil-Volberg [23]. The L^p case is by Hytönen [8]. An easier proof of the L^p case using interpolation and the L^2 case is by Lacey-Vähäkangas [14]. We state the L^p -version adapted to our situation here:

4.13. Lemma. *Let $s \in (1, \infty)$ and $\omega_0 \in \Omega$. There exists a constant $\mathcal{P}_B(\sigma, s)$, where σ is the goodness parameter, such that*

$$\mathcal{P}_B(\sigma, s) \rightarrow 0, \quad \text{as } \sigma \rightarrow \infty,$$

and

$$\mathbb{E}_\omega \left\| \sum_{\substack{Q \in \mathcal{D}_0(\omega_0) \\ Q \text{ is } \omega\text{-bad}}} \Delta_Q^1 f \right\|_{L^s(\mu)} \leq \mathcal{P}_B(\sigma, s) \|f\|_{L^s(\mu)}, \quad \text{for every } f \in L^s(\mu),$$

where \mathbb{E}_ω denotes the expectation over $\omega \in \Omega$.

When it is clear from the context, we also denote by $\mathbb{E}_\omega := \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} \mathbb{E}_{\omega_3}$ the expectation over $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega \times \Omega \times \Omega$. The result of Lemma 4.13 can directly be extended to the case of three lattices. Indeed, suppose $f \in L^s(\mu)$. Since a (ω_2, ω_3) -bad cube is either ω_2 -bad or ω_3 -bad, we have

$$\left(\sum_{\substack{Q \in \mathcal{D}_0(\omega_1) \\ Q \text{ is } (\omega_2, \omega_3)\text{-bad}}} |\Delta_Q^1 f|^2 \right)^{1/2} \leq \left(\sum_{\substack{Q \in \mathcal{D}_0(\omega_1) \\ Q \text{ is } \omega_2\text{-bad}}} |\Delta_Q^1 f|^2 \right)^{1/2} + \left(\sum_{\substack{Q \in \mathcal{D}_0(\omega_1) \\ Q \text{ is } \omega_3\text{-bad}}} |\Delta_Q^1 f|^2 \right)^{1/2}.$$

Thus, there holds by (4.12) that

$$\begin{aligned} \mathbb{E}_\omega \|P_B^1(\omega)f\|_{L^s(\mu)} &\lesssim \mathbb{E}_\omega \left\| \sum_{\substack{Q \in \mathcal{D}_0(\omega_1) \\ Q \text{ is } \omega_2\text{-bad}}} \Delta_Q^1 f \right\|_{L^s(\mu)} + \mathbb{E}_\omega \left\| \sum_{\substack{Q \in \mathcal{D}_0(\omega_1) \\ Q \text{ is } \omega_3\text{-bad}}} \Delta_Q^1 f \right\|_{L^s(\mu)} \\ &= \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} \left\| \sum_{\substack{Q \in \mathcal{D}_0(\omega_1) \\ Q \text{ is } \omega_2\text{-bad}}} \Delta_Q^1 f \right\|_{L^s(\mu)} + \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_3} \left\| \sum_{\substack{Q \in \mathcal{D}_0(\omega_1) \\ Q \text{ is } \omega_3\text{-bad}}} \Delta_Q^1 f \right\|_{L^s(\mu)} \\ &\leq 2\mathcal{P}_B(\sigma, s) \|f\|_{L^s(\mu)}, \end{aligned}$$

where we applied Lemma 4.13 in the last step. The same conclusion holds of course with the operators $P_B^2(\omega)$ and $P_B^3(\omega)$.

Observe also that

$$(4.14) \quad \|P_G^i(\omega)f\|_{L^s(\mu)} \lesssim \|f\|_{L^s(\mu)}$$

uniformly for $\omega \in \Omega \times \Omega \times \Omega$. This follows directly from Equation (4.12).

. Having introduced the b -adapted martingales and the good and bad cubes, we continue with the proof of Theorem 4.2 from Equation (4.8). Consider three functions $f \in L^p(\mu)$, $g \in L^q(\mu)$ and $h \in L^{r'}(\mu)$ with supports in $Q_0 \setminus Q_{0,\partial}$ and with

norm at most one. For every $\omega \in \Omega \times \Omega \times \Omega$ we can split $\langle T_{\mu,\Phi}(f, g), h \rangle_\mu$, without denoting the dependence on ω , as

$$\begin{aligned} & \langle T_{\mu,\Phi}(P_B^1 f, g), h \rangle_\mu + \langle T_{\mu,\Phi}(P_G^1 f, P_B^2 g), h \rangle_\mu \\ & + \langle T_{\mu,\Phi}(P_G^1 f, P_G^2 g), P_B^3 h \rangle_\mu + \langle T_{\mu,\Phi}(P_G^1 f, P_G^2 g), P_G^3 h \rangle_\mu. \end{aligned}$$

On average the absolute value of the terms where there is at least one bad function involved is small. Indeed, applying Lemma 4.13 and Equation (4.14) we have for example that

$$\begin{aligned} & \left| \mathbb{E}_\omega \langle T_{\mu,\Phi}(P_G^1(\omega) f, P_G^2(\omega) g), P_B^3(\omega) h \rangle_\mu \right| \\ & \leq \mathbb{E}_\omega \|T_{\mu,\Phi}\| \|P_G^1(\omega) f\|_{L^p(\mu)} \|P_G^2(\omega) g\|_{L^q(\mu)} \|P_B^3(\omega) h\|_{L^{r'}(\mu)} \\ & \lesssim \|T_{\mu,\Phi}\| \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \mathbb{E}_\omega \|P_B^3(\omega) h\|_{L^{r'}(\mu)} \\ & \lesssim \mathcal{P}_B(\sigma, r') \|T_{\mu,\Phi}\|. \end{aligned}$$

Thus, there exists a constant C such that

$$(4.15) \quad |\langle T_{\mu,\Phi}(f, g), h \rangle_\mu| \leq C \mathcal{P}_B(\sigma) \|T_{\mu,\Phi}\| + \left| \mathbb{E}_\omega \langle T_{\mu,\Phi}(P_G^1 f, P_G^2 g), P_G^3 h \rangle_\mu \right|,$$

where $\mathcal{P}_B(\sigma) := \max(\mathcal{P}_B(\sigma, p), \mathcal{P}_B(\sigma, q), \mathcal{P}_B(\sigma, r'))$.

By fixing the goodness parameter σ to be big enough, there holds $C \mathcal{P}_B(\sigma) \leq 1/4$. Combining this with (4.8) we have shown that

$$\frac{\|T_{\mu,\Phi}\|}{4} \leq C + \sup_{f,g,h} \left| \mathbb{E}_\omega \langle T_{\mu,\Phi}(P_G^1(\omega) f, P_G^2(\omega) g), P_G^3(\omega) h \rangle_\mu \right|,$$

where the supremum is as in (4.8). Now we fix three functions f, g and h as in the supremum, and turn to proving that

$$(4.16) \quad \left| \mathbb{E}_\omega \langle T_{\mu,\Phi}(P_G^1(\omega) f, P_G^2(\omega) g), P_G^3(\omega) h \rangle_\mu \right| \leq C + \|T_{\mu,\Phi}\|/8.$$

Once this is done, the proof of Theorem 4.2 is complete.

Proof of (4.16). For $\omega \in \Omega \times \Omega \times \Omega$ define $f_\omega := P_G^1(\omega) f$, $g_\omega := P_G^2(\omega) g$ and $h_\omega := P_G^3(\omega) h$. Then there holds, for example, the identity

$$f_\omega = \sum_{Q \in \mathcal{D}_0(\omega_1)} \Delta_Q^1 f_\omega = \sum_{\substack{Q \in \mathcal{D}_0(\omega_1) \\ Q \text{ is } (\omega_2, \omega_3)\text{-good}}} \Delta_Q^1 f.$$

We have

$$\begin{aligned}
\langle T_{\mu,\Phi}(f_\omega, g_\omega), h_\omega \rangle_\mu &= \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(K) \leq \ell(I)}} \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(K) \leq \ell(J)}} \langle T_{\mu,\Phi}(\Delta_I^1 f_\omega, \Delta_J^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu \\
(4.17) \quad &+ \sum_{I \in \mathcal{D}_0(\omega_1)} \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(I) \leq \ell(J)}} \sum_{\substack{K \in \mathcal{D}_0(\omega_3) \\ \ell(I) < \ell(K)}} \langle T_{\mu,\Phi}^{1*}(\Delta_K^3 h_\omega, \Delta_J^2 g_\omega), \Delta_I^1 f_\omega \rangle_\mu \\
&+ \sum_{J \in \mathcal{D}_0(\omega_2)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(J) < \ell(I)}} \sum_{\substack{K \in \mathcal{D}_0(\omega_3) \\ \ell(J) < \ell(K)}} \langle T_{\mu,\Phi}^{2*}(\Delta_I^1 f_\omega, \Delta_K^3 h_\omega), \Delta_J^2 g_\omega \rangle_\mu.
\end{aligned}$$

These three triple sums are essentially symmetric. We will concentrate on the first one.

Suppose $K \in \mathcal{D}_0(\omega_3)$. The double sum $\sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(K) \leq \ell(I)}} \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(K) \leq \ell(J)}}$ can be organized as

$$\sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(K) \leq \ell(I)}} \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(I) \leq \ell(J)}} + \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(K) \leq \ell(J) < 2^{u_0}}} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(J) < \ell(I)}}.$$

Also, for $I \in \mathcal{D}_0(\omega_1)$ there holds

$$\sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(I) \leq \ell(J)}} \Delta_J^2 g_\omega = E_{\omega_2, \ell(I)/2}^2 g_\omega =: E_{\ell(I)/2}^2 g_\omega,$$

and for $J \in \mathcal{D}_0(\omega_2)$ with $\ell(J) < 2^{u_0}$ we have

$$\sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(J) < \ell(I)}} \Delta_I^1 f_\omega = E_{\omega_1, \ell(J)}^1 f_\omega =: E_{\ell(J)}^1 f_\omega.$$

Regarding the notation we make the following explanation. In what follows we always have some $\omega = (\omega_1, \omega_2, \omega_3)$ like here, and we understand that e.g. $E_{\omega_2, \ell(I)/2}^2 g_\omega = E_{\ell(I)/2}^2 g_\omega$ i.e. that the superscript 2 does not only mean that we use the function b_2 , but also that we use the dyadic grid $\mathcal{D}_0(\omega_2)$. With this understanding we may suppress the additional subscripts denoting the dyadic grid used. Now, combining the above with the bilinearity of $T_{\mu,\Phi}$ leads to

$$\begin{aligned}
&\sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(K) \leq \ell(I)}} \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(K) \leq \ell(J)}} \langle T_{\mu,\Phi}(\Delta_I^1 f_\omega, \Delta_J^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu \\
(4.18) \quad &= \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(K) \leq \ell(I)}} \langle T_{\mu,\Phi}(\Delta_I^1 f_\omega, E_{\ell(I)/2}^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu \\
&+ \sum_{\substack{K \in \mathcal{D}_0(\omega_3) \\ \ell(K) < 2^{u_0}}} \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(K) \leq \ell(J) < 2^{u_0}}} \langle T_{\mu,\Phi}(E_{\ell(J)}^1 f_\omega, \Delta_J^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu.
\end{aligned}$$

The two terms here are handled again essentially symmetrically, and for this reason we concentrate on the first.

The term we have reduced to so far is

$$(4.19) \quad \mathbb{E}_\omega \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(K) \leq \ell(I)}} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, E_{\ell(I)/2}^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu,$$

where we note that because of goodness of the functions involved there are only good cubes in the summations. For every two cubes $Q, R \subset \mathbb{R}^n$ define the number $d_{Q,R} := \max(2\sqrt{n}\ell(Q), \ell(Q)^\gamma \ell(R)^{1-\gamma})$. Related to the reduction into good cubes above, we can assume that the parameter σ is so large that $2\sqrt{n} \leq 2^{\sigma(1-\gamma)}$. In this case if $2^\sigma \ell(Q) \leq \ell(R)$, then $d_{Q,R} = \ell(Q)^\gamma \ell(R)^{1-\gamma}$. Thus, if $K \in \mathcal{D}_0(\omega_3)$ is ω_1 -good, $I \in \mathcal{D}_0(\omega_1)$, $2^\sigma \ell(K) \leq \ell(I)$ and $d(K, I) \leq d_{K,I}$, then by goodness $K \subset I$. The reason why we use the numbers $d_{Q,R}$ is that they sometimes ensure that there is enough separation to use Lemma 4.20.

Applying goodness, (4.19) can be written as a sum of the following three terms:

$$\begin{aligned} I &:= \mathbb{E}_\omega \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(K) \leq \ell(I) \\ d(K, I) > d_{K,I}}} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, E_{\ell(I)/2}^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu, \\ II &:= \mathbb{E}_\omega \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(K) \leq \ell(I) \leq 2^\sigma \ell(K) \\ d(K, I) \leq d_{K,I}}} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, E_{\ell(I)/2}^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu, \\ III &:= \mathbb{E}_\omega \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ 2^\sigma \ell(K) < \ell(I) \\ K \subset I}} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, E_{\ell(I)/2}^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu. \end{aligned}$$

Goodness was needed to conclude that if $K \in \mathcal{D}_0(\omega_3)$ is ω_1 -good and $I \in \mathcal{D}_0(\omega_1)$ is such that $2^\sigma \ell(K) < \ell(I)$ and $d(K, I) \leq d_{K,I} = \ell(K)^\gamma \ell(I)^{1-\gamma}$, then $K \subset I$. In the term II the average over $\omega \in \Omega \times \Omega \times \Omega$ will still be important, otherwise we just estimate uniformly for every given $\omega \in \Omega \times \Omega \times \Omega$.

An auxiliary estimate. Before going into the analysis of the above three parts, let us record an easy estimate that is useful in what follows.

4.20. Lemma. Suppose $A \subset \mathbb{R}^n$ is a bounded set and h_0 is a function supported on A such that $\int h_0 d\mu = 0$. Suppose also that $t \geq 2$ and $B \subset \mathbb{R}^{2n}$ is a set satisfying

$$B \subset \{(y, z) \in \mathbb{R}^{2n} : \inf_{x \in A} \max(|x - y|, |x - z|) \geq td(A)\}.$$

Then we have for $f_0, g_0 \in L_{loc}^1(\mu)$ that

$$|\langle \tilde{T}_{\mu, \Phi}(1_B f_0 \otimes g_0), h_0 \rangle_\mu| \lesssim \frac{1}{t^\alpha} \int M_{\mu, m}(f_0, g_0) |h_0| d\mu.$$

Proof. Applying the Hölder estimate in the x -variable, we have have for an arbitrary $x_A \in A$ that

$$\begin{aligned}
& |\langle \tilde{T}_{\mu, \Phi}(1_B f_0 \otimes g_0), h_0 \rangle_\mu | \\
&= \left| \iiint (K_\Phi(x, y, z) - K_\Phi(x_A, y, z)) 1_B(y, z) f_0(y) g_0(z) h_0(x) d\mu(x) d\mu(y) d\mu(z) \right| \\
&\lesssim \iiint \frac{d(A)^\alpha |f_0(y) g_0(z) h_0(x)|}{(td(A) + |x - y| + |x - z|)^{2m+\alpha}} d\mu(y) d\mu(z) d\mu(x) \\
&\lesssim \frac{1}{t^\alpha} \int M_{\mu, m}(f_0, g_0) |h_0| d\mu.
\end{aligned}$$

The last estimate used Lemma 1.3. \square

The separated sum. Now we begin with the term I . Fix some $\omega \in \Omega \times \Omega \times \Omega$. The sum over K is further divided into the sum over those $K \in \mathcal{D}_0(\omega_3)$ such that $\ell(K_0) < 2^{u_0}$ and those K with $\ell(K) = 2^{u_0}$.

Suppose first that $K \in \mathcal{D}_0(\omega_3)$ and $I \in \mathcal{D}_0(\omega_1)$ are such that $\ell(K) = \ell(I) = 2^{u_0}$ and $d(I, K) > \ell(K)^\gamma \ell(I)^{1-\gamma} = \ell(I)$. Then, applying directly the size condition of the kernel gives

$$|\langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, E_{\ell(I)/2}^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu| \lesssim \frac{\|\Delta_I^1 f_\omega\|_{L^1(\mu)} \|E_{\ell(I)/2}^2 g_\omega\|_{L^1(\mu)} \|\Delta_K^3 h_\omega\|_{L^1(\mu)}}{\ell(I)^{2m}}.$$

Since $\|E_{\ell(I)/2}^2 g_\omega\|_{L^1(\mu)} \lesssim \|g_\omega\|_{L^1(\mu)}$, summing over I and K leads to

$$\begin{aligned}
(4.21) \quad & \left| \sum_{\substack{K \in \mathcal{D}_0(\omega_3), I \in \mathcal{D}_0(\omega_1) \\ \ell(I) = \ell(K) = 2^{u_0} \\ d(K, I) > d_{K, I}}} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, E_{\ell(I)/2}^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu \right| \\
& \lesssim \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(I) = 2^{u_0}}} \|\Delta_I^1 f_\omega\|_{L^1(\mu)} \cdot \frac{\|g_\omega\|_{L^1(\mu)}}{2^{2u_0 m}} \cdot \sum_{\substack{K \in \mathcal{D}_0(\omega_3) \\ \ell(K) = 2^{u_0}}} \|\Delta_K^3 h_\omega\|_{L^1(\mu)} \\
& \lesssim \frac{\|f_\omega\|_{L^1(\mu)} \|g_\omega\|_{L^1(\mu)} \|h_\omega\|_{L^1(\mu)}}{2^{2u_0 m}}.
\end{aligned}$$

Hölder's inequality gives

$$\|f_\omega\|_{L^1(\mu)} \|g_\omega\|_{L^1(\mu)} \|h_\omega\|_{L^1(\mu)} \leq \mu(Q_0)^2 \|f_\omega\|_{L^p(\mu)} \|g_\omega\|_{L^q(\mu)} \|h_\omega\|_{L^{r'}(\mu)}.$$

Because $2^{u_0} \sim \ell(Q_0)$ there holds $\mu(Q_0)^2 / (2^{2u_0 m}) \lesssim 1$. Therefore, the left hand side of (4.21) can be estimated as

$$(4.22) \quad LHS(4.21) \lesssim \|f_\omega\|_{L^p(\mu)} \|g_\omega\|_{L^q(\mu)} \|h_\omega\|_{L^{r'}(\mu)} \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)}.$$

Now we turn to those K with $\ell(K) < 2^{u_0}$. In this case we know that $\int \Delta_K^3 h_\omega d\mu = 0$. Define for the moment the shorthand

$$\varphi_{K,l} := \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ 2^l \ell(K) = \ell(I) \\ d(K,I) > d_{K,I}}} \Delta_I^1 f_\omega,$$

where $K \in \mathcal{D}_0(\omega_3)$, $\ell(K) < 2^{u_0}$ and $l = 0, 1, \dots, u_0 - \log_2 \ell(K)$. Notice that $|\varphi_{K,l}| \leq |D_{2^l \ell(K)}^1 f_\omega|$, and that $d(\text{spt } \varphi_{K,l}, K) > d_{K,I}$. Lemma 4.20 gives

$$(4.23) \quad \left| \left\langle T_{\mu,\Phi}(\varphi_{K,l}, E_{2^{l-1}\ell(K)}^2 g_\omega), \Delta_K^3 h_\omega \right\rangle_\mu \right| \lesssim 2^{-l(1-\gamma)\alpha} \int M_{\mu,m}(D_{2^l \ell(K)}^1 f_\omega) M_{\mu,m} M_{\mu,\mathcal{D}_0(\omega_2)} g_\omega |\Delta_K^3 h_\omega| d\mu.$$

Next, we show that

$$(4.24) \quad \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{l=0}^{u_0 - \log_2 \ell(K)} 2^{-l(1-\gamma)\alpha} \int M_{\mu,m}(D_{2^l \ell(K)}^1 f_\omega) M_{\mu,m} M_{\mu,\mathcal{D}_0(\omega_2)} g_\omega |\Delta_K^3 h_\omega| d\mu \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)}.$$

The left hand side of (4.24) can be reorganized as

$$\begin{aligned} & \sum_{l=0}^{\infty} 2^{-l(1-\gamma)\alpha} \sum_{k: k \leq u_0 - l} \sum_{\substack{K \in \mathcal{D}_0(\omega_3) \\ \ell(K) = 2^k}} \int M_{\mu,m}(D_{2^{k+l}}^1 f_\omega) M_{\mu,m} M_{\mu,\mathcal{D}_0(\omega_2)} g_\omega |\Delta_K^3 h_\omega| d\mu \\ &= \sum_{l=0}^{\infty} 2^{-l(1-\gamma)\alpha} \sum_{k: k \leq u_0 - l} \int M_{\mu,m}(D_{2^{k+l}}^1 f_\omega) M_{\mu,m} M_{\mu,\mathcal{D}_0(\omega_2)} g_\omega |D_{2^k}^3 h_\omega| d\mu. \end{aligned}$$

Fix one $l \in \{0, 1, \dots\}$. We have

$$\begin{aligned} & \sum_{k: k \leq u_0 - l} \int M_{\mu,m}(D_{2^{k+l}}^1 f_\omega) M_{\mu,m} M_{\mu,\mathcal{D}_0(\omega_2)} g_\omega |D_{2^k}^3 h_\omega| d\mu \\ & \leq \left\| \left(\sum_{k: k \leq u_0 - l} M_{\mu,m}(D_{2^{k+l}}^1 f_\omega)^2 \right)^{1/2} \right\|_{L^p(\mu)} \|M_{\mu,m} M_{\mu,\mathcal{D}_0(\omega_2)} g_\omega\|_{L^q(\mu)} \\ & \quad \cdot \left\| \left(\sum_{k: k \leq u_0 - l} |D_{2^k}^3 h_\omega|^2 \right)^{1/2} \right\|_{L^{r'}(\mu)} \\ & \lesssim \|f_\omega\|_{L^p(\mu)} \|g_\omega\|_{L^q(\mu)} \|h_\omega\|_{L^{r'}(\mu)}, \end{aligned}$$

where we applied the Fefferman–Stein inequality for the radial maximal function $M_{\mu,m}$. Such a version of the Fefferman–Stein inequality follows e.g. by using that it is at least known to be true for non-homogeneous dyadic maximal functions, and dominating the radial maximal function $M_{\mu,m}$ by finitely many such dyadic

maximal functions. To prove (4.24) it only remains to sum the geometric series $\sum_{l \geq 0} 2^{-l(1-\gamma)\alpha}$.

Equations (4.23) and (4.24) combined show that for a fixed $\omega \in \Omega \times \Omega \times \Omega$ the part of I that consists of those K with $\ell(K) < 2^{u_0}$ satisfies the right bound. Since the estimates we have done have been independent of ω , this finishes the proof of

$$(4.25) \quad |I| \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)} \leq 1.$$

Deeply contained cubes; error terms. Here a part of the term III is considered. Again a uniform estimate will be made for every $\omega \in \Omega \times \Omega \times \Omega$. We fix one ω now until we have estimated the whole term III . The goal is to reduce the estimate to a so-called paraproduct that involves the function $T_{\mu, \Phi}(b_1, b_2)$, which will then allow us to apply the property $|T_{\mu, \Phi}(b_1, b_2)| \lesssim 1$. To achieve this, we must first estimate two error terms. The paraproduct is handled in the next subsection.

Let us first introduce some notation. Define

$$\begin{aligned} \mathcal{D}_h(\omega_3) &= \{K \in \mathcal{D}_0(\omega_3) : \ell(K) < 2^{u_0-\sigma}, \Delta_K^3 h_\omega \neq 0\} \\ &= \{K \in \mathcal{D}_0(\omega_3) : \ell(K) < 2^{u_0-\sigma}, K \text{ is } (\omega_1, \omega_2)\text{-good}, \Delta_K^3 h \neq 0\}. \end{aligned}$$

Suppose $K \in \mathcal{D}_h(\omega_3)$ and $l \in \mathbb{Z}$ are such that $2^\sigma \ell(K) \leq 2^l \ell(K) \leq 2^{u_0}$. Then, by the goodness of the cube K , there exist cubes $I \in \mathcal{D}(\omega_1)$ and $J \in \mathcal{D}(\omega_2)$ of side length $2^l \ell(K)$ containing K . Since moreover $\Delta_K^3 h \neq 0$, which implies

$$K \subset \{d(\cdot, \mathbb{R}^n \setminus Q_0) > \lambda_0 \ell(Q_0)/4\},$$

the above cubes actually satisfy $I \in \mathcal{D}_0(\omega_1)$ and $J \in \mathcal{D}_0(\omega_2)$. We denote these unique cubes I and J by $I_{K,l}$ and $J_{K,l}$, and sometimes also by $I(K, l)$ and $J(K, l)$. So, in this context “ I ” refers to the lattice $\mathcal{D}(\omega_1)$ and “ J ” refers to $\mathcal{D}(\omega_2)$. These definitions depend on ω , but it does not matter since it is fixed for the moment.

We write

$$\begin{aligned} & \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ 2^\sigma \ell(K) < \ell(I) \\ K \subset I}} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, E_{\ell(I)/2}^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu \\ &= \sum_{K \in \mathcal{D}_h(\omega_3)} \sum_{\substack{l \in \mathbb{Z} : \\ 2^\sigma \ell(K) < 2^l \ell(K) \leq 2^{u_0}}} \langle T_{\mu, \Phi}(\Delta_{I_{K,l}}^1 f_\omega, E_{2^{l-1}\ell(K)}^2 g_\omega), \Delta_K^3 h \rangle_\mu. \end{aligned}$$

Notice that $\Delta_{I_{K,l}}^1 f_\omega \otimes E_{2^{l-1}\ell(K)}^2 g_\omega$ can be written as the sum

$$1_{I_{K,l-1} \times J_{K,l-1}} \Delta_{I_{K,l}}^1 f_\omega \otimes E_{2^{l-1}\ell(K)}^2 g_\omega + 1_{(I_{K,l-1} \times J_{K,l-1})^c} \Delta_{I_{K,l}}^1 f_\omega \otimes E_{2^{l-1}\ell(K)}^2 g_\omega,$$

where further

$$\begin{aligned} 1_{I_{K,l-1} \times J_{K,l-1}} \Delta_{I_{K,l}}^1 f_\omega \otimes E_{2^{l-1}\ell(K)}^2 g_\omega &= (D_{I_{K,l-1}}^1 f_\omega) b_1 \otimes (E_{J_{K,l-1}}^2 g_\omega) b_2 \\ &\quad - 1_{(I_{K,l-1} \times J_{K,l-1})^c} (D_{I_{K,l-1}}^1 f_\omega) b_1 \otimes (E_{J_{K,l-1}}^2 g_\omega) b_2. \end{aligned}$$

Here $(D_{I_{K,l-1}}^1 f_\omega) b_1 \otimes (E_{J_{K,l-1}}^2 g_\omega) b_2$ leads to the main term, and the other two give the error terms.

The error terms can easily be handled using Lemma 4.20. Notice that

$$(4.26) \quad \left| \left\langle \tilde{T}_{\mu,\Phi} (1_{(I_{K,l-1} \times J_{K,l-1})^c} \Delta_{I_{K,l}}^1 f_\omega \otimes E_{2^{l-1}\ell(K)}^2 g_\omega), \Delta_K^3 h \right\rangle_\mu \right| \\ \lesssim 2^{-l(1-\gamma)\alpha} \int M_{\mu,m}(D_{2^l\ell(K)}^1 f_\omega) M_{\mu,m} M_{\mu,\mathcal{D}_0(\omega_2)} g_\omega |\Delta_K^3 h| d\mu$$

and

$$(4.27) \quad \left| \left\langle \tilde{T}_{\mu,\Phi} (1_{(I_{K,l-1} \times J_{K,l-1})^c} (D_{I_{K,l-1}}^1 f_\omega) b_1 \otimes (E_{J_{K,l-1}}^2 g_\omega) b_2), \Delta_K^3 h \right\rangle_\mu \right| \\ \lesssim 2^{-l(1-\gamma)\alpha} |D_{I_{K,l-1}}^1 f_\omega| |E_{J_{K,l-1}}^2 g_\omega| \int |\Delta_K^3 h| d\mu \\ \lesssim 2^{-l(1-\gamma)\alpha} \int |D_{2^l\ell(K)}^1 f_\omega| M_{\mu,\mathcal{D}_0(\omega_2)} g_\omega |\Delta_K^3 h| d\mu.$$

It is clear that the sum over K and l of these is bounded by $\|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)}$ – see (4.24).

Deeply contained cubes; paraproduct. We begin this subsection by proving the following lemma.

4.28. Lemma. *Let B be a function satisfying $|B| \lesssim 1$. Then for every $\omega \in \Omega$ and $\varphi \in L^r(\mu)$ we have*

$$(4.29) \quad \left\| \sum_{Q \in \mathcal{D}_0(\omega)} \langle \varphi \rangle_Q^\mu \Delta_Q^{1*} B \right\|_{L^r(\mu)} \lesssim \|\varphi\|_{L^r(\mu)}.$$

Proof. We begin by constructing the collection $\mathcal{S} \subset \mathcal{D}_0(\omega)$ of principal cubes for the function φ . Of course, this is a completely standard construction. Set

$$\mathcal{S}_0 := \{Q \in \mathcal{D}_0(\omega) : \ell(Q) = 2^{u_0}\},$$

and suppose $\mathcal{S}_0, \dots, \mathcal{S}_k$ are defined for some k . If $S \in \mathcal{S}_k$, we define $\text{ch}_{\mathcal{S}}(S)$ to be the maximal cubes $Q \in \mathcal{D}_0(\omega)$ such that $Q \subset S$ and

$$\langle |\varphi| \rangle_Q^\mu > 2 \langle |\varphi| \rangle_S^\mu.$$

Then, set $\mathcal{S}_{k+1} := \bigcup_{S \in \mathcal{S}_k} \text{ch}_{\mathcal{S}}(S)$, and finally $\mathcal{S} := \bigcup_{k=0}^\infty \mathcal{S}_k$. The collection \mathcal{S} is obviously a Carleson family of cubes with respect to the measure μ . For every $Q \in \mathcal{D}_0(\omega)$ there exists a cube $S \in \mathcal{S}$ such that $Q \subset S$; the minimal such S is denoted by $\pi_{\mathcal{S}} Q$. By the construction of \mathcal{S} there holds

$$\langle |\varphi| \rangle_Q^\mu \leq 2 \langle |\varphi| \rangle_{\pi_{\mathcal{S}} Q}^\mu, \quad Q \in \mathcal{D}_0(\omega).$$

Turning to (4.29), we can use the principal cubes together with (4.12) to have

$$(4.30) \quad \left\| \sum_{Q \in \mathcal{D}_0(\omega)} \langle \varphi \rangle_Q^\mu \Delta_Q^{1*} B \right\|_{L^r(\mu)} \lesssim \left\| \sum_{S \in \mathcal{S}} \langle |\varphi| \rangle_S^\mu \sum_{\substack{Q \in \mathcal{D}_0(\omega) \\ \pi_{\mathcal{S}} Q = S}} \Delta_Q^{1*} B \right\|_{L^r(\mu)}.$$

Suppose $S \in \mathcal{S}$, $S' \in \text{ch}_{\mathcal{S}}(S)$ and $x \in S'$. If $\ell(S) < 2^{u_0}$, there holds

$$\left| \sum_{\substack{Q \in \mathcal{D}_0(\omega) \\ \pi_{\mathcal{S}} Q = S}} \Delta_Q^{1*} B(x) \right| = \left| \frac{\langle b_1 B \rangle_{S'}^\mu}{\langle b_1 \rangle_{S'}^\mu} - \frac{\langle b_1 B \rangle_S^\mu}{\langle b_1 \rangle_S^\mu} \right| \lesssim 1,$$

while if $\ell(S) = 2^{u_0}$, then

$$\left| \sum_{\substack{Q \in \mathcal{D}_0(\omega) \\ \pi_{\mathcal{S}} Q = S}} \Delta_Q^{1*} B(x) \right| = \left| \frac{\langle b_1 B \rangle_{S'}^\mu}{\langle b_1 \rangle_{S'}^\mu} \right| \lesssim 1.$$

On the other hand, if $\ell(S) < 2^{u_0}$, for μ -almost every $x \in S \setminus \bigcup_{S' \in \text{ch}_{\mathcal{S}}(S)} S'$ we have

$$\left| \sum_{\substack{Q \in \mathcal{D}_0(\omega) \\ \pi_{\mathcal{S}} Q = S}} \Delta_Q^{1*} B(x) \right| = \left| B(x) - \frac{\langle b_1 B \rangle_S^\mu}{\langle b_1 \rangle_S^\mu} \right| \lesssim 1,$$

and if $\ell(S) = 2^{u_0}$, then

$$\left| \sum_{\substack{Q \in \mathcal{D}_0(\omega) \\ \pi_{\mathcal{S}} Q = S}} \Delta_Q^{1*} B(x) \right| = |B(x)| \lesssim 1.$$

Hence, the right hand side of (4.30) is dominated by

$$\left\| \sum_{S \in \mathcal{S}} \langle |\varphi| \rangle_S^\mu 1_S \right\|_{L^r(\mu)} \lesssim \|\varphi\|_{L^r(\mu)},$$

where we applied the Carleson embedding theorem in a form that appears at least in Theorem 2.2 of [26]. \square

We will aim to apply the previous lemma with $B = T_{\mu, \Phi}(b_1, b_2)$, $\omega = \omega_3$ and Δ_Q^{1*} replaced with Δ_Q^{3*} . However, we require some preparations to see the connection. So to simplify notation define $B = T_{\mu, \Phi}(b_1, b_2)$ recalling that $|B| \lesssim 1$. Next, recall that we are left with controlling

$$\sum_{K \in \mathcal{D}_h(\omega_3)} \sum_{l=\sigma+1}^{u_0 - \log_2 \ell(K)} D_{I_{K,l-1}}^1 f_\omega E_{J_{K,l-1}}^2 g_\omega \langle B, \Delta_K^3 h \rangle_\mu.$$

For technical reasons, we want to cut away the finitely many l for which $2^{l-1} \ell(K) \geq 2^{u_0 - \sigma}$. This will lead to an error term, but we first focus on the remaining term i.e.

$$\sum_{K \in \mathcal{D}_h(\omega_3)} \sum_{l=\sigma+1}^{u_0 - \log_2 \ell(K) - \sigma} D_{I_{K,l-1}}^1 f_\omega E_{J_{K,l-1}}^2 g_\omega \langle B, \Delta_K^3 h \rangle_\mu.$$

Next, we split this into two parts by adding and subtracting $E_{J_{K,l+\sigma}}^2 g_\omega$ to every term. We first focus on

$$\sum_{K \in \mathcal{D}_h(\omega_3)} \langle B, \Delta_K^3 h \rangle_\mu \sum_{l=\sigma+1}^{u_0 - \log_2 \ell(K) - \sigma} D_{I_{K,l-1}}^1 f_\omega E_{J_{K,l+\sigma}}^2 g_\omega,$$

which we consider to be the main term.

To simplify the above, we need to define an auxiliary function. Set

$$F_\omega = \sum_{\substack{I \in \mathcal{D}_0(\omega_1)_{\text{good}} \\ \ell(I) \leq 2^{u_0 - \sigma} \\ \Delta_I^1 f \neq 0}} E_{J_{I,\sigma}}^2 g_\omega \Delta_I^1 f.$$

Notice that for the I appearing in the summation the cube $J_{I,\sigma} \in \mathcal{D}_0(\omega_2)$ always exists. Let us calculate $D_{I_{K,l-1}}^1 F_\omega$ for a fixed K as in the above summation and for a fixed $l \in \{\sigma + 1, \dots, u_0 - \log_2 \ell(K) - \sigma\}$. Notice that $\ell(I_{K,l}) \leq 2^{u_0 - \sigma}$. So, if in addition, $I_{K,l}$ is good and $\Delta_{I_{K,l}}^1 f \neq 0$, then

$$\Delta_{I_{K,l}} F_\omega = E_{J(I_{K,l}, \sigma)}^2 g_\omega \Delta_{I_{K,l}}^1 f = E_{J(I_{K,l}, \sigma)}^2 g_\omega \Delta_{I_{K,l}}^1 f_\omega.$$

This implies that in this case

$$D_{I_{K,l-1}}^1 F_\omega = E_{J(I_{K,l}, \sigma)}^2 g_\omega D_{I_{K,l-1}}^1 f_\omega.$$

But notice that both $J(I_{K,l}, \sigma)$ and $J_{K,l+\sigma}$ belong to $\mathcal{D}_0(\omega_2)$, are of common side length and contain K . Therefore, they are the same cube and we have

$$D_{I_{K,l-1}}^1 F_\omega = E_{J_{K,l+\sigma}}^2 g_\omega D_{I_{K,l-1}}^1 f_\omega.$$

If $I_{K,l}$ is not such a cube (i.e. it is bad or $\Delta_{I_{K,l}}^1 f \equiv 0$), then $D_{I_{K,l-1}}^1 F_\omega = 0 = D_{I_{K,l-1}}^1 f_\omega$. Therefore, the above identity holds and makes sense (as $J_{K,l+\sigma}$ is well-defined here) for every $l \in \{\sigma + 1, \dots, u_0 - \log_2 \ell(K) - \sigma\}$. We conclude that

$$\sum_{l=\sigma+1}^{u_0 - \log_2 \ell(K) - \sigma} D_{I_{K,l-1}}^1 f_\omega E_{J_{K,l+\sigma}}^2 g_\omega = \sum_{l=\sigma+1}^{u_0 - \log_2 \ell(K) - \sigma} D_{I_{K,l-1}}^1 F_\omega = E_{I_{K,\sigma}}^1 F_\omega.$$

In this telescoping identity we also used that by the definition of F_ω we have

$$E_{I_{K, u_0 - \log_2 \ell(K) - \sigma}}^1 F_\omega = 0,$$

since $I_{K, u_0 - \log_2 \ell(K) - \sigma} \in \mathcal{D}_0(\omega_1)$ satisfies $\ell(I_{K, u_0 - \log_2 \ell(K) - \sigma}) = 2^{u_0 - \sigma}$.

Our main term is now simplified to

$$\sum_{\substack{K \in \mathcal{D}_h(\omega_3) \\ \ell(K) \leq 2^{u_0 - 2\sigma - 1}}} \langle B, \Delta_K^3 h \rangle_\mu E_{I_{K,\sigma}}^1 F_\omega = \left\langle \sum_{\substack{K \in \mathcal{D}_h(\omega_3) \\ \ell(K) \leq 2^{u_0 - 2\sigma - 1}}} E_{I_{K,\sigma}}^1 F_\omega \Delta_K^{3*} B, h \right\rangle_\mu.$$

In what follows it will be useful to notice that

$$|E_{I_{K,\sigma}}^1 F_\omega| \lesssim \langle |F_\omega| \rangle_{I_{K,\sigma}}^\mu \leq \langle M_{\mu, \mathcal{D}_0(\omega_1)} F_\omega \rangle_K^\mu.$$

Using Hölder's inequality and (4.12) we have

$$\left| \left\langle \sum_{\substack{K \in \mathcal{D}_h(\omega_3) \\ \ell(K) \leq 2^{u_0-2\sigma-1}}} E_{I_{K,\sigma}}^1 F_\omega \Delta_K^{3*} B, h \right\rangle_\mu \right| \lesssim \left\| \sum_{K \in \mathcal{D}_0(\omega_3)} \langle M_{\mu, \mathcal{D}_0(\omega_1)} F_\omega \rangle_K^\mu \Delta_K^{3*} B \right\|_{L^r(\mu)} \|h\|_{L^{r'}(\mu)}.$$

Using Lemma 4.28 we see that

$$\left\| \sum_{K \in \mathcal{D}_0(\omega_3)} \langle M_{\mu, \mathcal{D}_0(\omega_1)} F_\omega \rangle_K^\mu \Delta_K^{3*} B \right\|_{L^r(\mu)} \lesssim \|F_\omega\|_{L^r(\mu)},$$

where we have using (4.12) and Hölder's inequality again that

$$\begin{aligned} \|F_\omega\|_{L^r(\mu)} &\lesssim \left\| \left(\sum_{\substack{I \in \mathcal{D}_0(\omega_1)_{\text{good}} \\ \ell(I) \leq 2^{u_0-\sigma} \\ \Delta_I^1 f \neq 0}} |E_{J_{I,\sigma}}^2 g_\omega|^2 |\Delta_I^1 f|^2 \right)^{1/2} \right\|_{L^r(\mu)} \\ &\lesssim \left\| \left(\sum_{I \in \mathcal{D}_0(\omega_1)} |\Delta_I^1 f|^2 \right)^{1/2} M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega \right\|_{L^r(\mu)} \\ &\leq \left\| \left(\sum_{I \in \mathcal{D}_0(\omega_1)} |\Delta_I^1 f|^2 \right)^{1/2} \right\|_{L^p(\mu)} \|M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega\|_{L^q(\mu)} \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}. \end{aligned}$$

We have now controlled a major part of

$$\sum_{K \in \mathcal{D}_h(\omega_3)} \sum_{l=\sigma+1}^{u_0 - \log_2 \ell(K)} D_{I_{K,l-1}}^1 f_\omega E_{J_{K,l-1}}^2 g_\omega \langle B, \Delta_K^3 h \rangle_\mu.$$

Tracking back what we did above, we see that we still need to control the following two error terms:

$$\begin{aligned} E_1 &= \sum_{\substack{K \in \mathcal{D}_h(\omega_3) \\ \ell(K) \leq 2^{u_0-2\sigma-1}}} \langle B, \Delta_K^3 h \rangle_\mu \sum_{l=\sigma+1}^{u_0 - \log_2 \ell(K) - \sigma} D_{I_{K,l-1}}^1 f_\omega (E_{J_{K,l-1}}^2 g_\omega - E_{J_{K,l+\sigma}}^2 g_\omega), \\ E_2 &= \sum_{K \in \mathcal{D}_h(\omega_3)} \langle B, \Delta_K^3 h \rangle_\mu \sum_{l=\max(\sigma+1, u_0 - \log_2 \ell(K) - \sigma + 1)}^{u_0 - \log_2 \ell(K)} D_{I_{K,l-1}}^1 f_\omega E_{J_{K,l-1}}^2 g_\omega. \end{aligned}$$

We control E_1 first. Recall that

$$S^1 f_\omega = \left(\sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(I) < 2^{u_0}}} |D_I^1 f_\omega|^2 1_I \right)^{1/2},$$

and define the following variant of this

$$S_\sigma^2 g_\omega = \left(\sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(J) < 2^{u_0 - \sigma}}} |E_J^2 g_\omega - E_{J(\sigma+1)}^2 g_\omega|^2 1_J \right)^{1/2}.$$

Notice that

$$\begin{aligned} & \left| \sum_{l=\sigma+1}^{u_0 - \log_2 \ell(K) - \sigma} D_{I_{K,l-1}}^1 f_\omega (E_{J_{K,l-1}}^2 g_\omega - E_{J_{K,l+\sigma}}^2 g_\omega) \right| \\ & \leq \left(\sum_{l=\sigma+1}^{u_0 - \log_2 \ell(K) - \sigma} |D_{I_{K,l-1}}^1 f_\omega|^2 \right)^{1/2} \left(\sum_{l=\sigma+1}^{u_0 - \log_2 \ell(K) - \sigma} |E_{J_{K,l-1}}^2 g_\omega - E_{J_{K,l+\sigma}}^2 g_\omega|^2 \right)^{1/2} \\ & \leq \inf_{x \in K} S^1 f_\omega(x) S_\sigma^2 g_\omega(x) \leq \langle S^1 f_\omega S_\sigma^2 g_\omega \rangle_K^\mu. \end{aligned}$$

Notice also that

$$\|S^1 f_\omega S_\sigma^2 g_\omega\|_{L^r(\mu)} \leq \|S^1 f_\omega\|_{L^p(\mu)} \|S_\sigma^2 g_\omega\|_{L^q(\mu)} \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

Here we used that also S_σ^2 is bounded – this is trivial by observing that $S_\sigma^2 g$ can be dominated pointwise by $(\sigma + 1)S^2 g$. After these observations E_1 is controlled exactly as the main term above – simply replace $E_{I_{K,\sigma}}^1 F_\omega$ by

$$p_K := \sum_{l=\sigma+1}^{u_0 - \log_2 \ell(K) - \sigma} D_{I_{K,l-1}}^1 f_\omega (E_{J_{K,l-1}}^2 g_\omega - E_{J_{K,l+\sigma}}^2 g_\omega),$$

and the used upper bound $\langle M_{\mu, \mathcal{D}_0(\omega_1)} F_\omega \rangle_K^\mu$ by $\langle S^1 f_\omega S_\sigma^2 g_\omega \rangle_K^\mu$.

It remains to deal with the term E_2 . Define

$$v_K = \sum_{l=\max(\sigma+1, u_0 - \log_2 \ell(K) - \sigma + 1)}^{u_0 - \log_2 \ell(K)} D_{I_{K,l-1}}^1 f_\omega E_{J_{K,l-1}}^2 g_\omega.$$

Notice that there is at most σ different l here for a given K as in E_2 . For this reason, we can make the following rough estimate:

$$|v_K| \lesssim \langle S^1 f_\omega M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega \rangle_K^\mu.$$

Since it is obvious that

$$\|S^1 f_\omega M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega\|_{L^r(\mu)} \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)},$$

the very same argument as above again yields that $|E_2| \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)}$.

Putting everything together from this subsection and the previous, we have shown that for every $\omega \in \Omega \times \Omega \times \Omega$ there holds

$$\left| \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ 2^\sigma \ell(K) < \ell(I) \\ K \subset I}} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, E_{\ell(I)/2}^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu \right| \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)},$$

and so also

$$(4.31) \quad |III| \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)} \leq 1.$$

The diagonal. Here we need to deal with the final term

$$(4.32) \quad II = \mathbb{E}_\omega \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(K) \leq \ell(I) \leq 2^\sigma \ell(K) \\ d(K, I) \leq d_{K, I}}} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, E_{\ell(I)/2}^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu.$$

For two cubes Q and R in \mathbb{R}^n let us write $Q \sim R$ if $\ell(Q) \leq \ell(R) \leq 2^\sigma \ell(Q)$ and $d(Q, R) \leq d_{Q, R}$. Notice the non-symmetry of this definition related to the side-lengths.

Notice that

$$II = \mathbb{E}_\omega \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ K \sim I}} \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(J) \geq \ell(I)}} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, \Delta_J^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu.$$

We divide the argument into several steps.

Step 1. If $\omega \in \Omega$ we agree that $\mathcal{D}'_0(\omega)$ denotes the cubes $Q \in \mathcal{D}_0(\omega)$ with $\ell(Q) < 2^{u_0}$. In this step we control

$$II_1 = \mathbb{E}_\omega \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ K \sim I}} \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(J) \geq \ell(I) \\ d(J, K) > d_{K, J}}} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, \Delta_J^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu,$$

which we further split as

$$II'_1 = \mathbb{E}_\omega \sum_{K \in \mathcal{D}'_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ K \sim I}} \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(J) \geq \ell(I) \\ d(J, K) > d_{K, J}}} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, \Delta_J^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu$$

and

$$II''_1 = \mathbb{E}_\omega \sum_{\substack{K \in \mathcal{D}_0(\omega_3) \\ \ell(K) = 2^{u_0}}} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ K \sim I}} \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(J) \geq \ell(I) \\ d(J, K) > d_{K, J}}} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, \Delta_J^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu.$$

This will be done by estimating uniformly in $\omega \in \Omega \times \Omega \times \Omega$.

We handle II_1'' first. Notice that here we have $\ell(I) = \ell(J) = \ell(K) = 2^{u_0}$ and $d_{K,J} > 2^{u_0}$. Therefore, it holds

$$\begin{aligned} |II_1''| &\leq \mathbb{E}_\omega \sum_{\substack{K \in \mathcal{D}_0(\omega_3) \\ \ell(K)=2^{u_0}}} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ \ell(I)=2^{u_0}}} \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(J)=2^{u_0}}} \frac{\|\Delta_I^1 f_\omega\|_{L^1(\mu)} \|\Delta_J^2 g_\omega\|_{L^1(\mu)} \|\Delta_K^3 h_\omega\|_{L^1(\mu)}}{2^{2mu_0}} \\ &\lesssim \mathbb{E}_\omega \frac{\|f_\omega\|_{L^1(\mu)} \|g_\omega\|_{L^1(\mu)} \|h_\omega\|_{L^1(\mu)}}{2^{2mu_0}} \\ &\leq \mathbb{E}_\omega \frac{\mu(Q_0)^2}{2^{2mu_0}} \|f_\omega\|_{L^p(\mu)} \|g_\omega\|_{L^q(\mu)} \|h_\omega\|_{L^{r'}(\mu)} \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)}, \end{aligned}$$

where we used that $2^{u_0} \sim \ell(Q_0)$. This finishes the estimate for the term II_1'' .

We now consider II_1' . Fix $l \geq 0$ and write

$$\phi_{K,I,l} := \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ d(J,K) > d_{K,I} \\ \ell(J)=2^l \ell(I)}} \Delta_J^2 g_\omega.$$

We can agree that this function vanishes for those l for which $2^l \ell(I) > 2^{u_0}$. Notice that $|\phi_{K,I,l}| \leq |D_{2^l \ell(I)}^2 g_\omega| \lesssim M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega$. Using Lemma 4.20 we see that here

$$\begin{aligned} &|\langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, \phi_{K,I,l}), \Delta_K^3 h_\omega \rangle_\mu| \\ &\lesssim 2^{-l(1-\gamma)\alpha} \int M_{\mu,m}(\Delta_I^1 f_\omega) M_{\mu,m}(M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega) |\Delta_K^3 h_\omega| d\mu. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\sum_{l \geq 0} \sum_{K \in \mathcal{D}'_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ K \sim I}} |\langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, \phi_{K,I,l}), \Delta_K^3 h_\omega \rangle_\mu| \\ &\lesssim \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ K \sim I}} \int M_{\mu,m}(\Delta_I^1 f_\omega) M_{\mu,m}(M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega) |\Delta_K^3 h_\omega| d\mu \\ &\lesssim \left\| \left(\sum_{I \in \mathcal{D}_0(\omega_1)} M_{\mu,m}(\Delta_I^1 f_\omega)^2 \right)^{1/2} \right\|_{L^p(\mu)} \|M_{\mu,m}(M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega)\|_{L^q(\mu)} \\ &\quad \times \left\| \left(\sum_{K \in \mathcal{D}_0(\omega_3)} |\Delta_K^3 g_\omega|^2 \right)^{1/2} \right\|_{L^{r'}(\mu)} \\ &\lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)}, \end{aligned}$$

where we used that given I there are only $\lesssim 1$ cubes K so that $K \sim I$, and the usual bounds for maximal functions and square functions. We have now controlled II_1' , and so are done with Step 1:

$$(4.33) \quad |II_1| \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)} \leq 1.$$

Step 2. In this subsection we bound

$$II_2 = \mathbb{E}_\omega \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ K \sim I}} \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(I) \leq \ell(J) \leq 2^\sigma \ell(K) \\ d(J, K) \leq d_{K, J}}} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, \Delta_J^2 g_\omega, \Delta_K^3 h_\omega) \rangle_\mu.$$

To ease the notation, let us define $\tilde{\mathfrak{D}}(\omega)$ to be the collection of triples

$$(I, J, K) \in \mathcal{D}_0(\omega_1) \times \mathcal{D}_0(\omega_2) \times \mathcal{D}_0(\omega_3)$$

such that $K \sim I$, $\ell(I) \leq \ell(J) \leq 2^\sigma \ell(K)$ and $d(J, K) \leq d_{K, J}$. Then, define $\mathfrak{D}(\omega)$ to be those triples (I', J', K') such that there exists $(I, J, K) \in \tilde{\mathfrak{D}}(\omega)$ so that $I' \in \text{ch}(I)$, $J' \in \text{ch}(J)$ and $K' \in \text{ch}(K)$. Notice that

$$(4.34) \quad II_2 = \mathbb{E}_\omega \sum_{(I, J, K) \in \mathfrak{D}(\omega)} \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_I b_1, D_J^2 g_\omega 1_J b_2), D_K^3 h_\omega 1_K b_3 \rangle_\mu.$$

We will decompose

$$\langle T_{\mu, \Phi}(1_I b_1, 1_J b_2), 1_K b_3 \rangle_\mu$$

using surgery for the triple of cubes (I, J, K) .

Surgery for a triple $(I, J, K) \in \mathfrak{D}(\omega)$. We perform surgery on $(I, J, K) \in \mathfrak{D}(\omega)$ with a parameter $\theta > 0$. Let $j(\theta) \in \mathbb{Z}$ be such that $2^{-21}\theta \leq 2^{j(\theta)} < 2^{-20}\theta$. Let $\mathcal{D}(\omega_4)$ be yet another random grid in \mathbb{R}^n , independent of all other grids considered. Define $\mathcal{Q} := \{Q \in \mathcal{D}(\omega_4) : \ell(Q) = 2^{j(\theta)} \ell(K)\}$, and for $x \in \mathbb{R}^n$, let $Q(x)$ be the unique cube in \mathcal{Q} that contains x . We define

$$\begin{aligned} I_\partial^{J, K}(\omega_4) &= I_\partial := \{x \in I : d(Q(x), \partial J) < \theta \ell(J)/2\} \\ &\quad \cup \{x \in I : d(Q(x), \partial K) < \theta \ell(K)/2\} \\ &\quad \cup \{x \in I \cap J \cap K : d(x, \partial Q(x)) < \theta \ell(Q(x))\}. \end{aligned}$$

Thus points in I_∂ belong to I , and are either close to the boundary of J , to the boundary of K , or to the boundary of the grid \mathcal{Q} . The set I_∂ depends on the cubes J and K . However, we have

$$(4.35) \quad \begin{aligned} I_\partial \subset I_{\text{bad}}^{\omega_2, \omega_3, \omega_4} &:= \bigcup_{\substack{J' \in \mathcal{D}(\omega_2) \\ \ell(J') \sim \ell(I)}} \{x \in I : d(x, \partial J') < \theta \ell(J')\} \\ &\quad \cup \bigcup_{\substack{K' \in \mathcal{D}(\omega_3) \\ \ell(K') \sim \ell(I)}} \{x \in I : d(x, \partial K') < \theta \ell(K')\} \\ &\quad \cup \bigcup_{\substack{Q \in \mathcal{D}(\omega_4) \\ \ell(Q) \sim 2^{j(\theta)} \ell(I)}} \{x \in I : d(x, \partial Q) < \theta \ell(Q)\}, \end{aligned}$$

which depends only on I and $\omega_2, \omega_3, \omega_4$.

We set

$$I_{\text{sep}}^{J, K}(\omega_4) = I_{\text{sep}} := I \setminus (I_\partial \cup (J \cap K)),$$

the part of I strictly separated from either J or K . Finally, we have

$$I_{\Delta}^{J,K}(\omega_4) = I_{\Delta} := I \setminus (I_{\partial} \cup I_{\text{sep}}) = \bigcup_i L_{I,i},$$

where each $L_{I,i}$ is of the form $L_{I,i} = (1 - \theta)Q \cap I \cap J \cap K$ for some $Q \in \mathcal{Q}$, and $\#i \lesssim_{\theta} 1$.

We have the partition

$$I = I_{\text{sep}} \cup I_{\partial} \cup I_{\Delta} = I_{\text{sep}} \cup I_{\partial} \cup \bigcup_i L_{I,i},$$

and in a completely analogous manner also

$$J = J_{\text{sep}} \cup J_{\partial} \cup J_{\Delta} = J_{\text{sep}} \cup J_{\partial} \cup \bigcup_j L_{J,j}$$

and

$$K = K_{\text{sep}} \cup K_{\partial} \cup K_{\Delta} = K_{\text{sep}} \cup K_{\partial} \cup \bigcup_k L_{K,k}.$$

A key observation is that all $L_{I,i} \subset I \cap J \cap K$ appearing in the first union are cubes (of the form $(1 - \theta)Q$ for $Q \in \mathcal{Q}$) unless they are close to ∂I , and they are never close to the boundary of J or K . A similar statement is valid for the cubes appearing in the other unions (related to J or K), and therefore it follows that if a cube L appears in all of the above unions (or just in two of them), then $L = (1 - \theta)Q$ for some $Q \in \mathcal{Q}$. Such cubes also satisfy $5L \subset I \cap J \cap K$ by construction.

. We now continue with Step 2. We fix $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega \times \Omega \times \Omega$ and $\omega_4 \in \Omega$. We will find an upper bound for

$$II_2(\omega) := \sum_{(I,J,K) \in \mathfrak{D}(\omega)} \langle T_{\mu,\Phi}(D_I^1 f_{\omega} 1_I b_1, D_J^2 g_{\omega} 1_J b_2), D_K^3 h_{\omega} 1_K b_3 \rangle_{\mu}$$

that depends on these random parameters. We will then take expectations.

Let $(I, J, K) \in \mathfrak{D}(\omega)$, and let $I = I_{\text{sep}} \cup I_{\partial} \cup I_{\Delta}$ be the decomposition from above. Since every $y \in I_{\text{sep}}$ satisfies $\max(d(y, J), d(y, K)) \gtrsim_{\theta} \ell(I)$, we have

$$|\langle T_{\mu,\Phi}(D_I^1 f_{\omega} 1_{I_{\text{sep}}} b_1, D_J^2 g_{\omega} 1_J b_2), D_K^3 h_{\omega} 1_K b_3 \rangle_{\mu}| \lesssim |D_I^1 f_{\omega} D_J^2 g_{\omega} D_K^3 h_{\omega}| \frac{\mu(I)\mu(J)\mu(K)}{\ell(I)^{2m}}.$$

Notice that

$$|D_I^1 f_{\omega} D_J^2 g_{\omega}| \frac{\mu(I)\mu(J)}{\ell(I)^{2m}} \lesssim M_{\mu,m}(D_I^1 f_{\omega} 1_I)(x) M_{\mu,m}(D_J^2 g_{\omega} 1_J)(x)$$

for every $x \in K$, since the cubes are of comparable size and close to each other. Hence

$$\begin{aligned} \sum_{(I,J,K) \in \mathfrak{D}(\omega)} |D_I^1 f_\omega D_J^2 g_\omega D_K^3 h_\omega| \frac{\mu(I)\mu(J)\mu(K)}{\ell(I)^{2m}} \\ \lesssim \sum_{(I,J,K) \in \mathfrak{D}(\omega)} \int M_{\mu,m}(D_I^1 f_\omega 1_I) M_{\mu,m}(D_J^2 g_\omega 1_J) |D_K^3 h_\omega| 1_K d\mu, \end{aligned}$$

which is dominated via Hölder's inequality by

$$\begin{aligned} \left\| \left(\sum_{(I,J,K) \in \mathfrak{D}(\omega)} M_{\mu,m}(D_I^1 f_\omega 1_I)^2 \right)^{1/2} \right\|_{L^p(\mu)} \left\| M_{\mu,m}(M_{\mu,\mathcal{D}_0(\omega_2)} g_\omega) \right\|_{L^q(\mu)} \\ \cdot \left\| \left(\sum_{(I,J,K) \in \mathfrak{D}(\omega)} |D_K^3 h_\omega|^2 1_K \right)^{1/2} \right\|_{L^{r'}(\mu)}. \end{aligned}$$

Since for every $I \in \mathcal{D}_0(\omega_1)$ the number of triples such that $(I, J, K) \in \mathfrak{D}(\omega)$ is uniformly bounded, and similarly for every $K \in \mathcal{D}_0(\omega_3)$, this last expression is in turn dominated by $\|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)}$. We have now controlled

$$\sum_{(I,J,K) \in \mathfrak{D}(\omega)} \langle T_{\mu,\Phi}(D_I^1 f_\omega 1_{I_{\text{sep}}} b_1, D_J^2 g_\omega 1_J b_2), D_K^3 h_\omega 1_K b_3 \rangle_\mu$$

uniformly in $\omega \in \Omega \times \Omega \times \Omega$ and $\omega_4 \in \Omega$ (recall that here $I_{\text{sep}} = I_{\text{sep}}^{J,K}(\omega_4)$).

. We will next control the sum, where I_{sep} is replaced with I_∂ . The point is to estimate this using the a priori boundedness of $T_{\mu,\Phi}$, and apply the fact that on average (with respect to ω_2, ω_3 and ω_4) the function $\left(\sum_I |D_I^1 f|^2 1_{I_{\text{bad}}^{\omega_2, \omega_3, \omega_4}} \right)^{1/2}$ has a small norm. The a priori boundedness can be utilised via the next lemma.

4.36. Lemma. *For a sequence $(f_i)_{i \in \mathbb{Z}}$ of functions and $s \in (1, \infty)$ define*

$$\|(f_i)_{i \in \mathbb{Z}}\|_{L^s(\mu; \ell^2)} := \left\| \left(\sum_{i \in \mathbb{Z}} |f_i|^2 \right)^{1/2} \right\|_{L^s(\mu)}.$$

If $(f_i)_i \subset L^p(\mu)$, $(g_i)_i \subset L^q(\mu)$ and $(h_i)_i \subset L^{r'}(\mu)$ are sequences of functions, then

$$\begin{aligned} \left| \sum_i \langle T_{\mu,\Phi}(f_i, g_i), h_i \rangle_\mu \right| \\ \lesssim \|T_{\mu,\Phi}\| \|(f_i)_i\|_{L^p(\mu; \ell^2)} \|(g_i)_i\|_{L^q(\mu; \ell^2)} \|(h_i)_i\|_{L^{r'}(\mu; \ell^2)}. \end{aligned}$$

This statement is in the spirit of a classical theorem by Marcinkiewicz and Zygmund. We recall a quick proof using random signs.

Proof of Lemma 4.36. Let $(f_i)_i \subset L^p(\mu)$, $(g_i)_i \subset L^q(\mu)$ and $(h_i)_i \subset L^{r'}(\mu)$. We may suppose that only finitely many functions in these sequences are non-zero. Let $\{\varepsilon_i\}_i$ and $\{\varepsilon'_i\}_i$ be two collections of independent random signs and denote by \mathbb{E} and \mathbb{E}' the related expectations, correspondingly.

Taking the random signs into use we have

$$\sum_i \langle T_{\mu, \Phi}(f_i, g_i), h_i \rangle_\mu = \mathbb{E} \left\langle \sum_i \varepsilon_i T_{\mu, \Phi}(f_i, g_i), \sum_k \varepsilon_k h_k \right\rangle,$$

where further

$$\sum_i \varepsilon_i T_{\mu, \Phi}(f_i, g_i) = \mathbb{E}' T_{\mu, \Phi} \left(\sum_i \varepsilon_i \varepsilon'_i f_i, \sum_j \varepsilon'_j g_j \right).$$

Thus, if we write $F_\varepsilon := \sum_i \varepsilon_i \varepsilon'_i f_i$, $G_\varepsilon := \sum_j \varepsilon'_j g_j$ and $H_\varepsilon := \sum_k \varepsilon_k h_k$, then

$$\begin{aligned} \left| \sum_i \langle T_{\mu, \Phi}(f_i, g_i), h_i \rangle_\mu \right| &= \left| \mathbb{E} \mathbb{E}' \langle T_{\mu, \Phi}(F_\varepsilon, G_\varepsilon), H_\varepsilon \rangle_\mu \right| \\ &\leq \mathbb{E} \mathbb{E}' \|T_{\mu, \Phi}\| \|F_\varepsilon\|_{L^p(\mu)} \|G_\varepsilon\|_{L^q(\mu)} \|H_\varepsilon\|_{L^{r'}(\mu)}, \end{aligned}$$

which by Hölder's inequality is at most

$$\|T_{\mu, \Phi}\| (\mathbb{E}' \|F_\varepsilon\|_{L^p(\mu)}^p)^{1/p} (\mathbb{E}' \|G_\varepsilon\|_{L^q(\mu)}^q)^{1/q} (\mathbb{E}' \|H_\varepsilon\|_{L^{r'}(\mu)}^{r'})^{1/r'}.$$

Applying Khintchine inequality there holds for example that

$$\mathbb{E} \mathbb{E}' \|F_\varepsilon\|_{L^p(\mu)}^p = \mathbb{E} \int \mathbb{E}' \left| \sum_i \varepsilon_i \varepsilon'_i f_i \right|^p d\mu \sim \int \left(\sum_i |f_i|^2 \right)^{p/2} d\mu.$$

Doing the same computation for G_ε and H_ε proves the lemma. \square

With help of Lemma 4.36 we have

$$\begin{aligned} &\left| \sum_{(I, J, K) \in \mathfrak{D}(\omega)} \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_{I_\partial}, D_J^2 g_\omega 1_J), D_K^3 h_\omega 1_K \rangle_\mu \right| \\ &\lesssim \|T_{\mu, \Phi}\| \left\| \left(\sum_{(I, J, K) \in \mathfrak{D}(\omega)} |D_I^1 f_\omega 1_{I_\partial}|^2 \right)^{1/2} \right\|_{L^p(\mu)} \left\| \left(\sum_{(I, J, K) \in \mathfrak{D}(\omega)} |D_J^2 g_\omega 1_J|^2 \right)^{1/2} \right\|_{L^q(\mu)} \\ &\quad \times \left\| \left(\sum_{(I, J, K) \in \mathfrak{D}(\omega)} |D_K^3 h_\omega 1_K|^2 \right)^{1/2} \right\|_{L^{r'}(\mu)} \\ &\lesssim \|T_{\mu, \Phi}\| \left\| \left(\sum_{I \in \mathcal{D}'_0(\omega_1)} |D_I^1 f 1_{I_{\text{bad}}^{\omega_2, \omega_3, \omega_4}}|^2 \right)^{1/2} \right\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)}. \end{aligned}$$

This is of the right form so that we can take averages at the end. Therefore, we now move on to the term where I_∂ is replaced with I_Δ .

. We have arrived at the term

$$\sum_{(I, J, K) \in \mathfrak{D}(\omega)} \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_{I_\Delta} b_1, D_J^2 g_\omega 1_J b_2), D_K^3 h_\omega 1_K b_3 \rangle_\mu.$$

Continuing in the natural way, we can dominate $II_2(\omega)$ with the sum of

$$C(\theta) \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)},$$

$$\|T_{\mu,\Phi}\| \left\| \left(\sum_{I \in \mathcal{D}'_0(\omega_1)} |D_I^1 f 1_{I_{\text{bad}}^{\omega_2, \omega_3, \omega_4}}|^2 \right)^{1/2} \right\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)},$$

two corresponding terms where the “bad” square function appears in g or h , and

$$(4.37) \quad \left| \sum_{(I,J,K) \in \mathfrak{D}(\omega)} \langle T_{\mu,\Phi}(D_I^1 f 1_{I_\Delta} b_1, D_J^2 g 1_{J_\Delta} b_2), D_K^3 h 1_{K_\Delta} b_3 \rangle_\mu \right|.$$

Suppose $(I, J, K) \in \mathfrak{D}(\omega)$. We further split $I_\Delta = \bigcup_i L_{I,i}$, $J_\Delta = \bigcup_j L_{J,j}$ and $K_\Delta = \bigcup_k L_{K,k}$. If (i, j, k) is such that $L_{I,i}$, $L_{J,j}$ and $L_{K,k}$ are not all equal, then separation between two of these is $\gtrsim_\theta \ell(I)$, and we have

$$|\langle T_{\mu,\Phi}(1_{L_{I,i}}, 1_{L_{J,j}}), 1_{L_{K,k}} \rangle_\mu| \lesssim_\theta \frac{\mu(L_{I,i})\mu(L_{J,j})\mu(L_{K,k})}{\ell(I)^{2m}} \lesssim \mu(I \cap J \cap K).$$

On the other hand, if (i, j, k) is such that $L_{I,i} = L_{J,j} = L_{K,k} =: L$, then by the observations made during the construction of the surgery, we have that L is a cube and $5L \subset I \cap J \cap K$. In this case we can use the weak boundedness assumption to have

$$|\langle T_{\mu,\Phi}(1_L, 1_L), 1_L \rangle_\mu| \lesssim \mu(5L) \leq \mu(I \cap J \cap K).$$

Since there are only $\lesssim_\theta 1$ cubes in the splittings of I_Δ , J_Δ and K_Δ , we have shown that

$$(4.38) \quad |\langle T_{\mu,\Phi}(1_{I_\Delta}, 1_{J_\Delta}), 1_{K_\Delta} \rangle_\mu| \lesssim_\theta \mu(I \cap J \cap K).$$

Using (4.38), we have

$$\begin{aligned} (4.37) &\lesssim_\theta \sum_{(I,J,K) \in \mathfrak{D}(\omega)} |D_I^1 f_\omega D_J^2 g_\omega D_K^3 h_\omega| \mu(I \cap J \cap K) \\ &\lesssim \left\| \left(\sum_{(I,J,K) \in \mathfrak{D}(\omega)} |D_I^1 f_\omega|^2 1_{I \cap J \cap K} \right)^{1/2} \right\|_{L^p(\mu)} \left\| \sup_{(I,J,K) \in \mathfrak{D}(\omega)} |D_J^2 g_\omega| 1_{I \cap J \cap K} \right\|_{L^q(\mu)} \\ &\quad \times \left\| \left(\sum_{(I,J,K) \in \mathfrak{D}(\omega)} |D_K^3 h_\omega|^2 1_{I \cap J \cap K} \right)^{1/2} \right\|_{L^{r'}(\mu)} \\ &\lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)}, \end{aligned}$$

where we used that $\sup_{(I,J,K) \in \mathfrak{D}(\omega)} |D_J^2 g_\omega| 1_{I \cap J \cap K} \lesssim M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega$.

Combining everything, and recalling that the functions have norm at most 1, we have shown that

$$\begin{aligned}
|II_2| &= |\mathbb{E}_\omega II_2(\omega)| = |\mathbb{E}_{\omega_4} \mathbb{E}_\omega II_2(\omega)| \\
&\lesssim C(\theta) + \|T_{\mu, \Phi}\| \mathbb{E}_{\omega_4} \mathbb{E}_\omega \left[\left\| \left(\sum_{I \in \mathcal{D}'_0(\omega_1)} |D_I^1 f|^2 1_{I_{\text{bad}}^{\omega_2, \omega_3, \omega_4}} \right)^{1/2} \right\|_{L^p(\mu)} \right. \\
&\quad + \left\| \left(\sum_{J \in \mathcal{D}'_0(\omega_2)} |D_J^2 g|^2 1_{J_{\text{bad}}^{\omega_1, \omega_3, \omega_4}} \right)^{1/2} \right\|_{L^q(\mu)} \\
&\quad \left. + \left\| \left(\sum_{K \in \mathcal{D}'_0(\omega_3)} |D_K^3 h|^2 1_{K_{\text{bad}}^{\omega_1, \omega_2, \omega_4}} \right)^{1/2} \right\|_{L^{r'}(\mu)} \right].
\end{aligned}$$

The above averages of bad square functions can be dominated with $c(\theta)$, where $\lim_{\theta \rightarrow 0} c(\theta) = 0$, as stated in Lemma 4.39. In L^2 this is easy and follows from the work of Nazarov–Treil–Volberg. In L^p such bounds have usually been obtained using some fairly heavy machinery (an improved contraction principle), see [8]. However, the interpolation technique used in [14] to control the good and bad parts of functions (as in Lemma 4.13) seems to lend an easier proof here also. For some reason this was not already used in [14], however. We record this simpler proof idea below.

4.39. Lemma. *The estimate*

$$(4.40) \quad \mathbb{E}_{\omega_2, \omega_3, \omega_4} \left\| \left(\sum_{I \in \mathcal{D}'_0(\omega_1)} |D_I^1 f|^2 1_{I_{\text{bad}}^{\omega_2, \omega_3, \omega_4}} \right)^{1/2} \right\|_{L^p(\mu)} \lesssim c(\theta, p) \|f\|_{L^p(\mu)}, \quad f \in L^p(\mu),$$

holds. Here $\lim_{\theta \rightarrow 0} c(\theta, p) = 0$.

Proof. For $f \in L^1_{\text{loc}}(\mu)$, define the operator \mathcal{S}_B by

$$\mathcal{S}_B f(x, \omega_2, \omega_3, \omega_4) := \left(\sum_{I \in \mathcal{D}'_0(\omega_1)} |D_I^1 f|^2 1_{I_{\text{bad}}^{\omega_2, \omega_3, \omega_4}}(x) \right)^{1/2}.$$

For a $\mu \times \mathbb{P} \times \mathbb{P} \times \mathbb{P}$ -measurable function φ write

$$\|\varphi\|_{L^p(\mu \times \mathbb{P} \times \mathbb{P} \times \mathbb{P})} := \left(\mathbb{E}_{\omega_2} \mathbb{E}_{\omega_3} \mathbb{E}_{\omega_4} \int |\varphi(x, \omega_2, \omega_3, \omega_4)|^p d\mu \right)^{1/p}.$$

We will show via interpolation that

$$\|\mathcal{S}_B f\|_{L^p(\mu \times \mathbb{P} \times \mathbb{P} \times \mathbb{P})} \lesssim c(\theta, p) \|f\|_{L^p(\mu)}.$$

This concludes the proof of the lemma, since the left hand side of (4.40) is at most $\|\mathcal{S}_B f\|_{L^p(\mu \times \mathbb{P} \times \mathbb{P} \times \mathbb{P})}$ by Hölder's inequality.

First, notice that trivially

$$\left(\sum_{I \in \mathcal{D}'_0(\omega_1)} |D_I^1 f|^2 1_{I_{\text{bad}}^{\omega_2, \omega_3, \omega_4}}(x) \right)^{1/2} \leq \left(\sum_{I \in \mathcal{D}'_0(\omega_1)} |D_I^1 f|^2 1_I(x) \right)^{1/2},$$

and so $\|\mathcal{S}_{\mathcal{B}}f\|_{L^p(\mu \times \mathbb{P} \times \mathbb{P} \times \mathbb{P})} \lesssim_p \|f\|_{L^p(\mu)}$, $p \in (1, \infty)$.

Considering the L^2 estimate, we have

$$\|\mathcal{S}_{\mathcal{B}}f\|_{L^2(\mu \times \mathbb{P} \times \mathbb{P} \times \mathbb{P})}^2 = \sum_{I \in \mathcal{D}'_0(\omega_1)} |D_I^1 f|^2 \mathbb{E}_{\omega_2} \mathbb{E}_{\omega_3} \mathbb{E}_{\omega_4} \mu(I_{\text{bad}}^{\omega_2, \omega_3, \omega_4}),$$

where it is standard that the average above is $\leq c(\theta)\mu(I)$ (see [23]). Using this it follows that

$$\|\mathcal{S}_{\mathcal{B}}f\|_{L^2(\mu \times \mathbb{P} \times \mathbb{P} \times \mathbb{P})}^2 \lesssim c(\theta) \|f\|_{L^2(\mu)}^2.$$

□

In synthesis, in Step 2 we have shown that

$$(4.41) \quad |II_2| \lesssim C(\theta) + c(\theta) \|T_{\mu, \Phi}\|.$$

We will not fix the parameter θ yet, since more surgeries will appear. Therefore, we are ready to move to Step 3.

Step 3. What is left after steps 1 and 2 is

$$II_3 = \mathbb{E}_{\omega} \sum_{K \in \mathcal{D}_0(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ K \sim I}} \sum_{\substack{J \in \mathcal{D}_0(\omega_2) \\ \ell(J) > 2^\sigma \ell(K) \\ d(J, K) \leq d_{K, J}}} \langle T_{\mu, \Phi}(\Delta_I^1 f_{\omega}, \Delta_J^2 g_{\omega}), \Delta_K^3 h_{\omega} \rangle_{\mu}.$$

Recall the set $\mathcal{D}_h(\omega_3)$ and the notation $J_{K, l} \in \mathcal{D}_0(\omega_2)$ for those $K \in \mathcal{D}_h(\omega_3)$ and $l \in \mathbb{Z}$ that satisfy $2^\sigma \ell(K) \leq 2^l \ell(K) \leq 2^{u_0}$. The existence of these cubes $J_{K, l}$ has been justified in the paraproduct section. We can now write

$$II_3 = \mathbb{E}_{\omega} \sum_{K \in \mathcal{D}_h(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ K \sim I}} \sum_{l: 2^\sigma \ell(K) < 2^l \ell(K) \leq 2^{u_0}} \langle T_{\mu, \Phi}(\Delta_I^1 f_{\omega}, \Delta_{J_{K, l}}^2 g_{\omega}), \Delta_K^3 h_{\omega} \rangle_{\mu}.$$

Next, we perform the standard splitting

$$\Delta_{J_{K, l}}^2 g_{\omega} = 1_{J_{K, l-1}^c} \Delta_{J_{K, l}}^2 g_{\omega} - (D_{J_{K, l-1}}^2 g_{\omega}) 1_{J_{K, l-1}^c} b_2 + (D_{J_{K, l-1}}^2 g_{\omega}) b_2.$$

Recalling that

$$\sum_{l: 2^\sigma \ell(K) < 2^l \ell(K) \leq 2^{u_0}} D_{J_{K, l-1}}^2 g_{\omega} = E_{J_{K, \sigma}}^2 g_{\omega}$$

we get, after replacing $\Delta_{J_{K, l}}^2 g_{\omega}$ with $(D_{J_{K, l-1}}^2 g_{\omega}) b_2$ in II_3 , the main term

$$II'_3 = \mathbb{E}_{\omega} \sum_{K \in \mathcal{D}_h(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ K \sim I}} \langle T_{\mu, \Phi}(\Delta_I^1 f_{\omega}, E_{J_{K, \sigma}}^2 g_{\omega} b_2), \Delta_K^3 h_{\omega} \rangle_{\mu}.$$

However, we first need to deal with the two error terms:

$$II''_3 = \mathbb{E}_{\omega} \sum_{K \in \mathcal{D}_h(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ K \sim I}} \sum_{l=\sigma+1}^{u_0 - \log_2 \ell(K)} \langle T_{\mu, \Phi}(\Delta_I^1 f_{\omega}, 1_{J_{K, l-1}^c} \Delta_{J_{K, l}}^2 g_{\omega}), \Delta_K^3 h_{\omega} \rangle_{\mu}$$

and

$$II_3''' = \mathbb{E}_\omega \sum_{K \in \mathcal{D}_h(\omega_3)} \sum_{\substack{I \in \mathcal{D}_0(\omega_1) \\ K \sim I}} \sum_{l=\sigma+1}^{u_0 - \log_2 \ell(K)} \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, (D_{J_{K,l-1}}^2 g_\omega) 1_{J_{K,l-1}^c} b_2), \Delta_K^3 h_\omega \rangle_\mu.$$

These are very brief to deal with as in the paraproduct section. Indeed, applying Lemma 4.20 we have

$$\begin{aligned} & \left| \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, 1_{J_{K,l-1}^c} \Delta_{J_{K,l}}^2 g_\omega), \Delta_K^3 h_\omega \rangle_\mu \right| \\ & \lesssim 2^{-l(1-\gamma)\alpha} \int M_{\mu, m}(\Delta_I^1 f_\omega) M_{\mu, m}(\Delta_{J_{K,l}}^2 g_\omega) |\Delta_K^3 h_\omega| d\mu \\ & \lesssim 2^{-l(1-\gamma)\alpha} \int M_{\mu, m}(\Delta_I^1 f_\omega) M_{\mu, m}(M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega) |\Delta_K^3 h_\omega| d\mu \end{aligned}$$

and similarly

$$\begin{aligned} & \left| \langle T_{\mu, \Phi}(\Delta_I^1 f_\omega, (D_{J_{K,l-1}}^2 g_\omega) 1_{J_{K,l-1}^c} b_2), \Delta_K^3 h_\omega \rangle_\mu \right| \\ & \lesssim 2^{-l(1-\gamma)\alpha} \int M_{\mu, m}(\Delta_I^1 f_\omega) |D_{J_{K,l-1}}^2 g_\omega| |\Delta_K^3 h_\omega| d\mu \\ & \lesssim 2^{-l(1-\gamma)\alpha} \int M_{\mu, m}(\Delta_I^1 f_\omega) M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega |\Delta_K^3 h_\omega| d\mu. \end{aligned}$$

These are summable to the right bound over the appropriate K, I and l , and so $|II_3''| + |II_3'''| \lesssim 1$.

We continue with the main term II_3' . Denote by $\tilde{\mathcal{D}}(\omega)$ those pairs (I, K) where $K \in \mathcal{D}_h(\omega_3)$, $I \in \mathcal{D}_0(\omega_1)$ and $K \sim I$. Then we define

$$\mathcal{D}(\omega) := \{(I', K') : I' \in \text{ch}(I) \text{ and } K' \in \text{ch}(K) \text{ for some } (I, K) \in \tilde{\mathcal{D}}(\omega)\}.$$

We can write

$$II_3' = \mathbb{E}_\omega \sum_{(I, K) \in \mathcal{D}(\omega)} \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_I b_1, (E_{J_{K(1), \sigma}}^2 g_\omega) b_2), D_K^3 h_\omega 1_K b_3 \rangle_\mu.$$

We shall perform another surgery argument, this time with the pairs $(I, K) \in \mathcal{D}(\omega)$. However, we are going to do this with the slight modification that the resulting cubes “ L ” in the intersection $I \cap K$ have small boundaries.

Surgery for a pair of cubes $(I, K) \in \mathcal{D}(\omega)$. We again perform surgery with a parameter $\theta > 0$, this time to a pair of cubes (I, K) . As before, we set $j(\theta) \in \mathbb{Z}$ so that $2^{-21}\theta \leq 2^{j(\theta)} < 2^{-20}\theta$, $\mathcal{Q} := \{Q \in \mathcal{D}(\omega_4) : \ell(Q) = 2^{j(\theta)}\ell(K)\}$ and let $Q(x)$ be the unique cube in \mathcal{Q} that contains x .

The small boundary consideration goes as follows. Let $Q \in \mathcal{Q}$ and consider the cube $(1 - \theta)Q$. By letting the small boundary parameter $t = t(\theta)$ to be large enough, there exists a cube S_Q , that is concentric with Q , satisfies $(1 - \theta)Q \subset S_Q \subset (1 - \theta/2)Q$ and has t -small boundary.

Now, we define the sets I_∂ , I_{sep} , I_Δ and I_{bad} in a natural way using the sets S_Q in place of $(1 - \theta)Q$:

$$I_\partial^K(\omega_4) = I_\partial := \{x \in I : d(Q(x), \partial K) < \theta \ell(K)/2\} \cup \{x \in I \cap K : x \notin S_{Q(x)}\};$$

$$\begin{aligned} I_\partial \subset I_{\text{bad}}^{\omega_3, \omega_4} &:= \bigcup_{\substack{K' \in \mathcal{D}(\omega_3) \\ \ell(K') \sim \ell(I)}} \{x \in I : d(x, \partial K') < \theta \ell(K')\} \\ &\cup \bigcup_{\substack{Q \in \mathcal{D}(\omega_4) \\ \ell(Q) \sim 2^{j(\theta)} \ell(I)}} \{x \in I : d(x, \partial Q) < \theta \ell(Q)\}; \end{aligned}$$

$$I_{\text{sep}}^K(\omega_4) = I_{\text{sep}} := I \setminus (I_\partial \cup K)$$

and

$$I_\Delta^K(\omega_4) = I_\Delta := I \setminus (I_\partial \cup I_{\text{sep}}) = \bigcup_i L_{I,i}.$$

Here each $L_{I,i}$ is of the form $S_Q \cap I \cap K$ for some $Q \in \mathcal{Q}$, $\#i \lesssim_\theta 1$, and $L_i = S_Q$ unless it is close to the boundary of I . As before, the same splitting is performed starting from the cube K , and one observes that if $L = L_{I,i} = L_{K,k}$, then L is a cube and $5L \subset I \cap K$.

. We now continue with II'_3 . We write $II'_3 = E_\omega II'_3(\omega) = E_{\omega_4} E_\omega II'_3(\omega)$, where

$$II'_3(\omega) = \sum_{(I,K) \in \mathcal{D}(\omega)} \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_I b_1, (E_{J_{K(1), \sigma}}^2 g_\omega) b_2), D_K^3 h_\omega 1_K b_3 \rangle_\mu.$$

We fix $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega \times \Omega \times \Omega$ and $\omega_4 \in \Omega$, and estimate $II'_3(\omega)$ with a bound depending on these random parameters.

First, we replace the cubes I with the separated parts I_{sep} . Let $(I, K) \in \mathcal{D}(\omega)$. Then

$$\begin{aligned} &|\langle T_{\mu, \Phi}(D_I^1 f_\omega 1_{I_{\text{sep}}} b_1, (E_{J_{K(1), \sigma}}^2 g_\omega) b_2), D_K^3 h_\omega 1_K b_3 \rangle_\mu| \\ &\lesssim_\theta \frac{|D_I^1 f_\omega| \mu(I) |E_{J_{K(1), \sigma}}^2 g_\omega| |D_K^3 h_\omega| \mu(K)}{\ell(I)^m}, \end{aligned}$$

where separation and the fact that

$$\int \frac{d\mu(z)}{(\ell(I) + |x - z|)^{2m}} \lesssim \ell(I)^{-m}$$

were used. Because $K^{(1)} \sim I^{(1)}$, there holds for every $x \in K$ that

$$\frac{|D_I^1 f_\omega| \mu(I)}{\ell(I)^m} \lesssim M_{\mu, m}(D_I^1 f_\omega 1_I)(x).$$

Since also $|E_{J_{K(1), \sigma}}^2 g_\omega| \lesssim M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega(x)$ for every $x \in K$ we have

$$\frac{|D_I^1 f_\omega| \mu(I) |E_{J_{K(1), \sigma}}^2 g_\omega| |D_K^3 h_\omega| \mu(K)}{\ell(I)^m} \lesssim \int M_{\mu, m}(D_I^1 f_\omega 1_I) M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega |D_K^3 h_\omega| 1_K d\mu.$$

This can be summed over $(I, K) \in \mathcal{D}(\omega)$ as we have seen many times.

Next, we look at the term that arises when in II'_3 the cubes I are replaced with I_∂ . As above with the other surgery argument, the point is to use the a priori boundedness of $T_{\mu, \Phi}$. To this end, we need the standard principal stopping cubes for the function g_ω in the grid $\mathcal{D}_0(\omega_2)$. These are constructed precisely as in the proof of Lemma 4.28. We denote this collection by $\mathcal{G} = \mathcal{G}(\omega)$.

For every $K \in \mathcal{D}_0(\omega_3)$ that appears in the pairs $(I, K) \in \mathcal{D}(\omega)$, define

$$a_K := \frac{E_{J_{K^{(1)}, \sigma}}^2 g_\omega}{\langle |g_\omega| \rangle_{\pi_{\mathcal{G}}(J_{K^{(1)}, \sigma})}^\mu}.$$

By the stopping condition (and accretivity) we know that $|a_K| \lesssim 1$. If $G \in \mathcal{G}(\omega)$, define $\mathcal{D}_G(\omega)$ to be those pairs $(I, K) \in \mathcal{D}(\omega)$ such that $\pi_{\mathcal{G}} J_{K^{(1)}, \sigma} = G$. Using the collections $\mathcal{D}_G(\omega)$ and the numbers a_K we have

$$\begin{aligned} (4.42) \quad & \sum_{(I, K) \in \mathcal{D}(\omega)} \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_{I_\partial} b_1, E_{J_{K^{(1)}, \sigma}}^2 g_\omega b_2), D_K^3 h_\omega 1_K b_3 \rangle_\mu \\ &= \sum_{G \in \mathcal{G}(\omega)} \langle |g_\omega| \rangle_G^\mu \sum_{(I, K) \in \mathcal{D}_G(\omega)} \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_{I_\partial} b_1, b_2), a_K D_K^3 h_\omega 1_K b_3 \rangle_\mu. \end{aligned}$$

Let $G \in \mathcal{G}(\omega)$ and $(I, K) \in \mathcal{D}_G(\omega)$. Using the goodness of $K^{(1)}$ and the size estimate of the kernel we have

$$|\langle T_{\mu, \Phi}(1_{I_\partial} b_1, 1_{G^c} b_2), 1_K b_3 \rangle_\mu| \lesssim \frac{\mu(I)\mu(K)}{(\ell(K)^\gamma \ell(G)^{1-\gamma})^m} \leq \frac{\mu(I)\mu(K)}{\ell(K)^m},$$

whence, using as before that $K^{(1)} \sim I^{(1)}$ and $K \subset G$, we have

$$\begin{aligned} & \langle |g_\omega| \rangle_G^\mu |\langle T_{\mu, \Phi}(D_I^1 f_\omega 1_{I_\partial} b_1, 1_{G^c} b_2), a_K D_K^3 h_\omega 1_K b_3 \rangle_\mu| \\ & \lesssim \int M_{\mu, m}(D_I^1 f_\omega 1_I) M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega |D_K^3 h_\omega| 1_K d\mu. \end{aligned}$$

Since this is summable over $(I, K) \in \mathcal{D}(\omega)$, to control the right hand side of (4.42) it suffices to estimate

$$(4.43) \quad \sum_{G \in \mathcal{G}(\omega)} \langle |g_\omega| \rangle_G^\mu \sum_{(I, K) \in \mathcal{D}_G(\omega)} \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_{I_\partial} b_1, 1_G b_2), a_K D_K^3 h_\omega 1_K b_3 \rangle_\mu.$$

Let $G \in \mathcal{G}(\omega)$. We enumerate $\mathcal{D}_G(\omega)$ by writing

$$\mathcal{D}_G(\omega) = \{(I, K)_i^G\}_i.$$

Let $(\varepsilon_i)_i$ be an independent sequence of random signs, and write \mathbb{E}_ε for the corresponding expectation. For the moment define the shorthands

$$f_{(I, K)} := D_I^1 f_\omega 1_{I_\partial} \quad \text{and} \quad h_{(I, K)} := a_K D_K^3 h_\omega 1_K, \quad (I, K) \in \mathcal{D}_G(\omega).$$

Applying this notation and the random signs there holds

$$\begin{aligned} & \sum_{(I,K) \in \mathcal{D}_G(\omega)} \langle T_{\mu,\Phi}(D_I^1 f_\omega 1_{I_\partial} b_1, 1_G b_2), a_K D_K^3 h_\omega 1_K b_3 \rangle_\mu \\ &= \sum_i \langle T_{\mu,\Phi}(f_{(I,K)_i^G}, 1_G b_2), h_{(I,K)_i^G} \rangle_\mu \\ &= \mathbb{E}_\varepsilon \langle T_{\mu,\Phi}(\psi_{\varepsilon,G}, 1_G b_2), \eta_{\varepsilon,G} \rangle_\mu, \end{aligned}$$

where

$$\psi_{\varepsilon,G} := \sum_i \varepsilon_i f_{(I,K)_i^G}$$

and

$$\eta_{\varepsilon,G} := \sum_i \varepsilon_i h_{(I,K)_i^G}.$$

Using this we can write (4.43) as

$$\sum_{G \in \mathcal{G}(\omega)} \langle |g_\omega| \rangle_G^\mu \mathbb{E}_\varepsilon \langle T_{\mu,\Phi}(\psi_{\varepsilon,G}, 1_G b_2), \eta_{\varepsilon,G} \rangle_\mu.$$

Now, Lemma 4.36 gives

$$\begin{aligned} |(4.43)| &\leq \mathbb{E}_\varepsilon \left\| \sum_{G \in \mathcal{G}(\omega)} \langle T_{\mu,\Phi}(\psi_{\varepsilon,G}, \langle |g_\omega| \rangle_G^\mu 1_G b_2), \eta_{\varepsilon,G} \rangle_\mu \right\| \\ &\lesssim \mathbb{E}_\varepsilon \|T_{\mu,\Phi}\| \left\| \left(\sum_{G \in \mathcal{G}(\omega)} |\psi_{\varepsilon,G}|^2 \right)^{1/2} \right\|_{L^p(\mu)} \left\| \left(\sum_{G \in \mathcal{G}(\omega)} (\langle |g_\omega| \rangle_G^\mu)^2 1_G \right)^{1/2} \right\|_{L^q(\mu)} \\ &\quad \times \left\| \left(\sum_{G \in \mathcal{G}(\omega)} |\eta_{\varepsilon,G}|^2 \right)^{1/2} \right\|_{L^{r'}(\mu)}. \end{aligned}$$

Carleson embedding theorem implies that the middle factor related to g_ω is dominated by $\|g\|_{L^q(\mu)}$. Concerning the other two factors, we have

$$\begin{aligned} & \mathbb{E}_\varepsilon \left\| \left(\sum_{G \in \mathcal{G}(\omega)} |\psi_{\varepsilon,G}|^2 \right)^{1/2} \right\|_{L^p(\mu)} \left\| \left(\sum_{G \in \mathcal{G}(\omega)} |\eta_{\varepsilon,G}|^2 \right)^{1/2} \right\|_{L^{r'}(\mu)} \\ &\leq \left(\mathbb{E}_\varepsilon \left\| \left(\sum_{G \in \mathcal{G}(\omega)} |\psi_{\varepsilon,G}|^2 \right)^{1/2} \right\|_{L^p(\mu)}^p \right)^{1/p} \left(\mathbb{E}_\varepsilon \left\| \left(\sum_{G \in \mathcal{G}(\omega)} |\eta_{\varepsilon,G}|^2 \right)^{1/2} \right\|_{L^{r'}(\mu)}^{r'} \right)^{1/r'}, \end{aligned}$$

and

$$\begin{aligned} (4.44) \quad & \mathbb{E}_\varepsilon \left\| \left(\sum_{G \in \mathcal{G}(\omega)} |\psi_{\varepsilon,G}|^2 \right)^{1/2} \right\|_{L^p(\mu)}^p = \mathbb{E}_\varepsilon \left\| \left(\sum_{G \in \mathcal{G}(\omega)} \left| \sum_i \varepsilon_i f_{(I,K)_i^G} \right|^2 \right)^{1/2} \right\|_{L^p(\mu)}^p \\ & \sim \left\| \left(\sum_{G \in \mathcal{G}(\omega)} \sum_i |f_{(I,K)_i^G}|^2 \right)^{1/2} \right\|_{L^p(\mu)}^p. \end{aligned}$$

Here we used an ℓ^2 -valued version of the Kahane–Khintchine inequality. Writing out the definition of the functions $f_{(I,K)}$ the right hand side of (4.44) can be estimated up by

$$\left\| \left(\sum_{I \in \mathcal{D}'_0(\omega_1)} |D_I^1 f_\omega|^2 1_{I_{\text{bad}}^{\omega_3, \omega_4}} \right)^{1/2} \right\|_{L^p(\mu)}^p.$$

The corresponding estimate holds for the functions $\eta_{\varepsilon, G}$, and the resulting $L^{r'}(\mu)$ norm of the square sum of the terms $|D_K^3 h_\omega| 1_K$ is $\lesssim \|h\|_{L^{r'}(\mu)}$.

Putting the above steps together, we have shown that

$$|(4.43)| \lesssim \|T_{\mu, \Phi}\| \left\| \left(\sum_{I \in \mathcal{D}'_0(\omega_1)} |D_I^1 f_\omega|^2 1_{I_{\text{bad}}^{\omega_3, \omega_4}} \right)^{1/2} \right\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)}.$$

We are left with the term

$$\sum_{(I, K) \in \mathcal{D}(\omega)} \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_{I_\Delta} b_1, (E_{J_{K(1), \sigma}}^2 g_\omega) b_2), D_K^3 h_\omega 1_K b_3 \rangle_\mu.$$

Repeating the arguments with K , we reduce to

$$(4.45) \quad \sum_{(I, K) \in \mathcal{D}(\omega)} \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_{I_\Delta} b_1, (E_{J_{K(1), \sigma}}^2 g_\omega) b_2), D_K^3 h_\omega 1_{K_\Delta} b_3 \rangle_\mu.$$

Let $(I, K) \in \mathcal{D}(\omega)$. We use the decompositions $I_\Delta = \bigcup_i L_{I,i}$ and $K_\Delta = \bigcup_k L_{K,k}$. If $L_{I,i} \neq L_{K,k}$, then the separation between these is $\gtrsim_\theta \ell(I)$, whence

$$\begin{aligned} & \left| \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_{L_{I,i}} b_1, (E_{J_{K(1), \sigma}}^2 g_\omega) b_2), D_K^3 h_\omega 1_{L_{K,k}} b_3 \rangle_\mu \right| \\ & \lesssim_\theta |D_I^1 f_\omega E_{J_{K(1), \sigma}}^2 g_\omega D_K^3 h_\omega| \frac{\mu(L_{I,i}) \mu(L_{K,k})}{\ell(I)^m} \\ & \lesssim |D_I^1 f_\omega E_{J_{K(1), \sigma}}^2 g_\omega D_K^3 h_\omega| \mu(I \cap K) \\ & \lesssim \int |D_I^1 f_\omega| 1_I M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega |D_K^3 h_\omega| 1_K d\mu. \end{aligned}$$

If $L_{I,i} = L_{K,k} = L$, then L is a cube that has a $t(\theta)$ -small boundary and $5L \subset I \cap K$. Using separation, we have

$$\begin{aligned} & \left| \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_L b_1, (E_{J_{K(1), \sigma}}^2 g_\omega) 1_{(2L)^c} b_2), D_K^3 h_\omega 1_L b_3 \rangle_\mu \right| \\ & \lesssim |D_I^1 f_\omega E_{J_{K(1), \sigma}}^2 g_\omega D_K^3 h_\omega| \frac{\mu(L) \mu(L)}{\ell(L)^m}, \end{aligned}$$

which produces the same bound as the estimate before this one.

We record the following Lemma at this point. This is just a minor modification of Lemma 9.44 in [24].

4.46. Lemma. *Let $t > 0$ and suppose Q is a cube with t -small boundary. Then*

$$|\langle T(1_Q, 1_{2Q \setminus Q}), 1_Q \rangle| \lesssim t \mu(2Q).$$

Proof. Size gives

$$|\langle T(1_Q, 1_{2Q \setminus Q}), 1_Q \rangle| \lesssim \iiint \frac{1_Q(y) 1_{2Q \setminus Q}(z) 1_Q(x)}{(|x-y| + |x-z|)^{2m}} d\mu(y) d\mu(z) d\mu(x).$$

Since Q has small boundary, the measure of the boundary is zero.

Fix for the moment some $z \in 2Q \setminus \overline{Q}$. Define

$$A_k := \{(x, y) \in Q \times Q : \max(|x-y|, |x-z|) \in [2^{k-1}d(z, Q), 2^k d(z, Q))\}.$$

The set A_k is non-empty only if $2^k d(z, Q) \leq C\ell(Q)$, which amounts to $k \leq \log_2(C\ell(Q)/d(z, Q))$. We can assume that $C > 2$. Also,

$$\begin{aligned} & \iint_{A_k} \frac{1_Q(y) 1_Q(x)}{(|x-y| + |x-z|)^{2m}} d\mu(y) d\mu(x) \\ & \leq (2^{k-1}d(z, Q))^{-2m} \int_{B(z, 2^{k+1}d(z, Q))} d\mu(y) \int_{B(z, 2^k d(z, Q))} d\mu(x) \lesssim 1. \end{aligned}$$

Hence there holds

$$\begin{aligned} & \iint \frac{1_Q(y) 1_Q(x)}{(|x-y| + |x-z|)^{2m}} d\mu(y) d\mu(x) \\ & = \sum_{k=1}^{\log_2(C\ell(Q)/d(z, Q))} \iint_{A_k} \frac{1_Q(y) 1_Q(x)}{(|x-y| + |x-z|)^{2m}} d\mu(y) d\mu(x) \\ & \lesssim \log_2(C\ell(Q)/d(z, Q)). \end{aligned}$$

Next, define

$$B_k := \{z \in 2Q : d(z, \partial Q) \in (2^{-k}C\ell(Q), 2^{-k+1}C\ell(Q)]\}, \quad k \in \{1, 2, \dots\}.$$

We have

$$\begin{aligned} \int_{2Q \setminus Q} \log_2(C\ell(Q)/d(z, Q)) d\mu(z) & \leq \sum_{k=1}^{\infty} k\mu(B_k) \\ & \leq \sum_{k=1}^{\infty} k \cdot t2^{-k+1}C\mu(2Q) \\ & \lesssim t\mu(2Q). \end{aligned}$$

□

Let us continue. Since L has a $t(\theta)$ -small boundary, we have by Lemma 4.46 that

$$\begin{aligned} & |\langle T_{\mu, \Phi}(D_I^1 f_\omega 1_L b_1, (E_{J_{K(1), \sigma}}^2 g_\omega) 1_{2L \setminus L} b_2), D_K^3 h_\omega 1_L b_3 \rangle_\mu| \\ & \lesssim_\theta |D_I^1 f_\omega E_{J_{K(1), \sigma}}^2 g_\omega D_K^3 h_\omega| \mu(2L). \end{aligned}$$

This again leads to the same estimate as above.

Finally, weak boundedness gives

$$\begin{aligned} & \left| \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_L b_1, (E_{J_{K^{(1)}, \sigma}}^2 g_\omega) 1_L b_2), D_K^3 h_\omega 1_L b_3 \rangle_\mu \right| \\ & \lesssim |D_I^1 f_\omega E_{J_{K^{(1)}, \sigma}}^2 g_\omega D_K^3 h_\omega|_\mu (5L), \end{aligned}$$

which again yields the same estimate.

Since there are $\lesssim_\theta 1$ cubes $L_{I,i}$ and $L_{K,k}$, we have shown that

$$\begin{aligned} & \left| \langle T_{\mu, \Phi}(D_I^1 f_\omega 1_{I_\Delta} b_1, (E_{J_{K^{(1)}, \sigma}}^2 g_\omega) b_2), D_K^3 h_\omega 1_{K_\Delta} b_3 \rangle_\mu \right| \\ & \lesssim_\theta \int |D_I^1 f_\omega| 1_I M_{\mu, \mathcal{D}_0(\omega_2)} g_\omega |D_K^3 h_\omega| 1_K d\mu. \end{aligned}$$

This in turn shows that

$$|(4.45)| \lesssim_\theta \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^{r'}(\mu)} \leq 1.$$

We are now ready with Step 3, since we have proved that

$$\begin{aligned} |II'_3| &= |E_{\omega_4} E_\omega II'_3(\omega)| \\ &\lesssim C(\theta) + \|T_{\mu, \Phi}\| \mathbb{E}_{\omega_4} \mathbb{E}_\omega \left[\left\| \left(\sum_{I \in \mathcal{D}'_0(\omega_1)} |D_I^1 f|^2 1_{I_{\text{bad}}^{\omega_2, \omega_3, \omega_4}} \right)^{1/2} \right\|_{L^p(\mu)} \right. \\ &\quad \left. + \left\| \left(\sum_{K \in \mathcal{D}'_0(\omega_3)} |D_K^3 h|^2 1_{K_{\text{bad}}^{\omega_1, \omega_2, \omega_4}} \right)^{1/2} \right\|_{L^{r'}(\mu)} \right]. \end{aligned}$$

As previously, this leads to the bound

$$|II'_3| \lesssim C(\theta) + c(\theta) \|T_{\mu, \Phi}\|$$

using Lemma 4.39. Therefore, we have shown that

$$(4.47) \quad |II_3| \lesssim C(\theta) + c(\theta) \|T_{\mu, \Phi}\|.$$

Synthesis of the diagonal. Taking into account (4.33), (4.41) and (4.47) we have shown that

$$|II| \lesssim C(\theta) + c(\theta) \|T_{\mu, \Phi}\|,$$

which implies that

$$(4.48) \quad |II| \leq C + \|T_{\mu, \Phi}\|/100$$

fixing θ small enough.

Synthesis. Taking into account (4.25), (4.31) and (4.48) (our bounds for I , II and III), and the various symmetric terms coming from the splittings (4.17) and (4.18), we have shown (4.16). This ends the proof. \square

5. END POINT ESTIMATES

Before proving the weak type bound we need to recall the non-homogeneous Calderón–Zygmund decomposition of measures, see [24].

5.1. Lemma. *Let μ be a Radon measure in \mathbb{R}^n . For every $\nu \in M(\mathbb{R}^n)$ with compact support and every $\lambda > 2^{n+1}\|\nu\|/\|\mu\|$, we have:*

- (1) *There exists a family of cubes $(Q_i)_i$ so that $\sum_i 1_{Q_i} \leq C_n$ and a function $f \in L^1(\mu)$ such that*

$$(5.2) \quad |\nu|(Q_i) > \frac{\lambda}{2^{n+1}}\mu(2Q_i),$$

$$(5.3) \quad |\nu|(\eta Q_i) \leq \frac{\lambda}{2^{n+1}}\mu(2\eta Q_i) \text{ for } \eta > 2,$$

$$(5.4) \quad \nu = f d\mu \text{ in } \mathbb{R}^n \setminus \bigcup_i Q_i, \text{ with } |f| \leq \lambda \mu\text{-a.e.}$$

- (2) *Suppose that for each i we are given a $(6, \beta_0)$ - μ -doubling cube R_i such that it is concentric with Q_i and $Q_i \subset R_i$. For each i set*

$$w_i = \frac{1_{Q_i}}{\sum_k 1_{Q_k}}.$$

Then there exists a family of functions $(\varphi_i)_i$ (of the form $\varphi_i = \alpha_i h_i$ for some constant $\alpha_i \in \mathbb{C}$ and non-negative function $h_i \geq 0$) such that

$$(5.5) \quad \text{spt } \varphi_i \subset R_i,$$

$$(5.6) \quad \int \varphi_i d\mu = \int w_i d\nu,$$

$$(5.7) \quad \sum_i |\varphi_i| \leq B\lambda \text{ (} B \text{ depends only on } \beta_0, n \text{),}$$

$$(5.8) \quad \|\varphi_i\|_{L^\infty(\mu)}\mu(R_i) \leq 2|\nu|(Q_i).$$

We are ready to prove the weak type bound. However, the next proposition is not enough, since we need the weak type bound for T_\sharp (the good lambda method requires this). To get this we shall need to combine the next proposition with a certain formulation of Cotlar's inequality and some additional arguments. We give the full details, since we are not aware of a reference covering this type of generality.

5.9. Proposition. *Let μ be a measure of order m on \mathbb{R}^n and T be a bilinear m -dimensional SIO. Let $1 < r, p, q < \infty$ be so that $1/p + 1/q = 1/r$, and suppose we have uniformly on $\varepsilon > 0$ that*

$$\|T_{\mu,\varepsilon}\|_{L^p(\mu) \times L^q(\mu) \rightarrow L^r(\mu)} \lesssim 1.$$

Then we have uniformly on $\varepsilon > 0$ that

$$\|T_\varepsilon\|_{M(\mathbb{R}^n) \times M(\mathbb{R}^n) \rightarrow L^{1/2,\infty}(\mu)} \lesssim 1.$$

Proof. We are given $\varepsilon > 0$, $\nu, \eta \in M(\mathbb{R}^n)$ and $\lambda > 0$, and want to prove that

$$\mu(\{x \in \mathbb{R}^n : |T_\varepsilon(\nu, \eta)(x)| > \lambda\}) \lesssim \left(\frac{\|\nu\| \|\eta\|}{\lambda} \right)^{1/2}.$$

Without loss of generality we can assume that $\|\nu\| = \|\eta\| = 1$. We can then also assume that $\lambda^{1/2} > 2^{n+1}/\|\mu\|$, since otherwise the claim is trivial.

Let us first assume that ν and η have compact support. Then we can perform the following decompositions using Lemma 5.1. Applying the lemma to the measure ν on the level $\lambda^{1/2}$ we get cubes $(Q_{1,i})_i$ and a function $f_1 \in L^1(\mu)$ like in (1) of Lemma 5.1. For each i let $R_{1,i}$ be the smallest $(6, 6^{m+1})$ - μ -doubling cube of the form $6^k Q_{1,i}$, $k \geq 0$. Such a cube exists by standard arguments, see [24]. Then let $w_{1,i}$ and $\varphi_{1,i}$ be like in (2) of Lemma 5.1. We write

$$\nu = f_1 d\mu + \sum_i \varphi_{1,i} d\mu + \sum_i (w_{1,i} d\nu - \varphi_{1,i} d\mu) =: g_1 d\mu + \sum_i \beta_{1,i},$$

where the function g_1 is defined by

$$g_1 = f_1 + \sum_i \varphi_{1,i}$$

and the complex measure $\beta_{1,i}$ is defined by

$$\beta_{1,i} = w_{1,i} d\nu - \varphi_{1,i} d\mu.$$

We also set

$$\beta_1 = \sum_i \beta_{1,i}.$$

It is easy to see the properties

$$(5.10) \quad \mu\left(\bigcup_i 2Q_{1,i}\right) \lesssim \frac{1}{\lambda^{1/2}},$$

$$\|g_1\|_{L^\infty(\mu)} \lesssim \lambda^{1/2}, \quad \|g_1\|_{L^1(\mu)} \lesssim 1 \text{ and}$$

$$(5.11) \quad \|g_1\|_{L^u(\mu)}^u \leq \|g\|_{L^\infty(\mu)}^{u-1} \|g\|_{L^1(\mu)} \lesssim (\lambda^{1/2})^{u-1}, \quad 1 < u < \infty.$$

Regarding the complex measures $\beta_{1,i}$ we have the following:

- (a) $\text{spt } \beta_{1,i} \subset R_{1,i}$;
- (b) $\beta_{1,i}(R_{1,i}) = 0$;
- (c) $\|\beta_{1,i}\| \leq 2|\nu|(Q_{1,i})$.

Finally, the fact that $R_{1,i}$ is the *smallest* $(6, 6^{m+1})$ - μ -doubling cube of the form $6^k Q_{1,i}$, $k \geq 0$, is utilised via the standard fact that it implies the estimate

$$(5.12) \quad \int_{R_{1,i} \setminus Q_{1,i}} \frac{d\mu(x)}{|x - c_{Q_{1,i}}|^m} \lesssim 1.$$

We then perform the exact same decomposition using the measure η . The notation for this decomposition is $Q_{2,j}$, $R_{2,j}$, f_2 , g_2 , $\varphi_{2,j}$ etc.

We will separately estimate the naturally appearing good–good, good–bad, bad–good, and bad–bad parts, denoted by I_{gg} , I_{gb} , I_{bg} and I_{bb} respectively. For example, we have

$$I_{gg} := \mu(\{x \in \mathbb{R}^n : T_{\mu,\varepsilon}(g_1, g_2)(x) > \lambda/4\}).$$

In fact, the estimate for I_{gg} is trivial using the assumed boundedness of $T_{\mu,\varepsilon}$ and (5.11):

$$I_{gg} \lesssim \lambda^{-r} [\|g_1\|_{L^p(\mu)} \|g_2\|_{L^q(\mu)}]^r \lesssim \lambda^{-r} (\lambda^{1/2})^{2r-1} = \lambda^{-1/2}.$$

Estimation of I_{bg} . We shall now estimate I_{bg} – the term I_{gb} is handled symmetrically. Because of (5.10) it is enough to estimate

$$\begin{aligned} & \mu\left(\left\{x \in \mathbb{R}^n \setminus \bigcup_i 2Q_{1,i} : |T_\varepsilon(\beta_{1,i}, g_2 d\mu)(x)| > \lambda/4\right\}\right) \\ & \lesssim \lambda^{-1} \sum_i \int_{\mathbb{R}^n \setminus 2Q_{1,i}} |T_\varepsilon(\beta_{1,i}, g_2 d\mu)(x)| d\mu(x). \end{aligned}$$

To control this it is enough to fix i and prove

$$(5.13) \quad \int_{\mathbb{R}^n \setminus 2Q_{1,i}} |T_\varepsilon(\beta_{1,i}, g_2 d\mu)(x)| d\mu(x) \lesssim \lambda^{1/2} |\nu|(Q_{1,i}).$$

We write $\int_{\mathbb{R}^n \setminus 2Q_{1,i}} = \int_{\mathbb{R}^n \setminus 2R_{1,i}} + \int_{2R_{1,i} \setminus 2Q_{1,i}}$, and estimate these separately.

We estimate the integral over $\mathbb{R}^n \setminus 2R_{1,i}$ first. Let $x \in \mathbb{R}^n \setminus 2R_{1,i}$. Notice that if $d(x, R_{1,i}) > \varepsilon$, then $\text{spt } \beta_{1,i} \subset R_{1,i} \subset B(x, \varepsilon)^c$. So if $d(x, R_{1,i}) > \varepsilon$, the fact that $\beta_{1,i}(R_{1,i}) = 0$ together with the y -continuity of K gives that

$$|T_\varepsilon(\beta_{1,i}, g_2 d\mu)(x)| \lesssim \lambda^{1/2} \frac{\ell(R_{1,i})^\alpha}{|x - c_{R_{1,i}}|^{m+\alpha}} |\nu|(Q_{1,i}),$$

where we also used that $\|g_2\|_{L^\infty(\mu)} \lesssim \lambda^{1/2}$ and $\|\beta_{1,i}\| \lesssim |\nu|(Q_{1,i})$. This shows that

$$\int_{\substack{\mathbb{R}^n \setminus 2R_{1,i} \\ d(x, R_{1,i}) > \varepsilon}} |T_\varepsilon(\beta_{1,i}, g_2 d\mu)(x)| d\mu(x) \lesssim \lambda^{1/2} |\nu|(Q_{1,i}).$$

The size estimate of K gives that

$$|T_\varepsilon(\beta_{1,i}, g_2 d\mu)(x)| \lesssim \lambda^{1/2} |\nu|(Q_{1,i}) \frac{1}{\varepsilon^m}.$$

Notice that if $x \in \mathbb{R}^n \setminus 2R_{1,i}$ satisfies $d(x, R_{1,i}) \leq \varepsilon$, then $|x - c_{R_{1,i}}| \lesssim d(x, R_{1,i}) \leq \varepsilon$. This gives that

$$\int_{\substack{\mathbb{R}^n \setminus 2R_{1,i} \\ d(x, R_{1,i}) \leq \varepsilon}} |T_\varepsilon(\beta_{1,i}, g_2 d\mu)(x)| d\mu(x) \lesssim \lambda^{1/2} |\nu|(Q_{1,i}) \frac{\mu(B(c_{R_{1,i}}, C\varepsilon))}{\varepsilon^m} \lesssim \lambda^{1/2} |\nu|(Q_{1,i}).$$

We have shown that

$$\int_{\mathbb{R}^n \setminus 2R_{1,i}} |T_\varepsilon(\beta_{1,i}, g_2 d\mu)(x)| d\mu(x) \lesssim \lambda^{1/2} |\nu|(Q_{1,i}),$$

and will next show that the same bound holds for

$$\int_{2R_{1,i} \setminus 2Q_{1,i}} |T_\varepsilon(\beta_{1,i}, g_2 d\mu)(x)| d\mu(x).$$

For this it is enough to separately bound

$$\int_{2R_{1,i} \setminus 2Q_{1,i}} |T_\varepsilon(w_{1,i} d\nu, g_2 d\mu)(x)| d\mu(x) \text{ and } \int_{2R_{1,i}} |T_{\mu,\varepsilon}(\varphi_{1,i}, g_2)(x)| d\mu(x).$$

We begin with the first integral. The size estimate gives

$$|T_\varepsilon(w_{1,i} d\nu, g_2 d\mu)(x)| \lesssim \lambda^{1/2} |\nu|(Q_{1,i}) \frac{1}{|x - c_{Q_{1,i}}|^m}, \quad x \notin 2Q_{1,i},$$

so we have by (5.12) that

$$\int_{2R_{1,i} \setminus 2Q_{1,i}} |T_\varepsilon(w_{1,i} d\nu, g_2 d\mu)(x)| d\mu(x) \lesssim \lambda^{1/2} |\nu|(Q_{1,i}).$$

Next, we bound

$$\begin{aligned} \int_{2R_{1,i}} |T_{\mu,\varepsilon}(\varphi_{1,i}, g_2)(x)| d\mu(x) &\leq \int_{2R_{1,i}} |T_{\mu,\varepsilon}(\varphi_{1,i}, 1_{4R_{1,i}} g_2)(x)| d\mu(x) \\ &\quad + \int_{2R_{1,i}} |T_{\mu,\varepsilon}(\varphi_{1,i}, 1_{(4R_{1,i})^c} g_2)(x)| d\mu(x). \end{aligned}$$

Using the assumed boundedness of $T_{\mu,\varepsilon}$ and recalling that $R_{1,i}$ is doubling we see that

$$\begin{aligned} \int_{2R_{1,i}} |T_{\mu,\varepsilon}(\varphi_{1,i}, 1_{4R_{1,i}} g_2)| d\mu &\lesssim \mu(R_{1,i})^{1-1/r} \cdot \|\varphi_{1,i}\|_{L^\infty(\mu)} \mu(R_{1,i})^{1/p} \cdot \lambda^{1/2} \mu(R_{1,i})^{1/q} \\ &= \lambda^{1/2} \|\varphi_{1,i}\|_{L^\infty(\mu)} \mu(R_{1,i}) \lesssim \lambda^{1/2} |\nu|(Q_{1,i}). \end{aligned}$$

We move on to bounding

$$\int_{2R_{1,i}} |T_{\mu,\varepsilon}(\varphi_{1,i}, 1_{(4R_{1,i})^c} g_2)(x)| d\mu(x).$$

Notice that for $x \in 2R_{1,i}$ we have

$$\int_{(4R_{1,i})^c} \int_{R_{1,i}} \frac{d\mu(y) d\mu(z)}{(|x-y| + |x-z|)^{2m}} \lesssim \mu(R_{1,i}) \int_{R_{1,i}^c} \frac{d\mu(z)}{|z - c_{R_{1,i}}|^{2m}} \lesssim \frac{\mu(R_{1,i})}{\ell(R_{1,i})^m} \lesssim 1,$$

so that

$$\int_{2R_{1,i}} |T_{\mu,\varepsilon}(\varphi_{1,i}, 1_{(4R_{1,i})^c} g_2)(x)| d\mu(x) \lesssim \lambda^{1/2} \|\varphi_{1,i}\|_{L^\infty(\mu)} \mu(R_{1,i}) \lesssim \lambda^{1/2} |\nu|(Q_{1,i}).$$

Putting everything together we have shown (5.13), which ends our treatment of the term I_{bg} .

Estimation of I_{bb} . Now we turn to estimate the final part

$$I_{bb} = \mu(\{x \in \mathbb{R}^n : |T_\varepsilon(\beta_1, \beta_2)(x)| > \lambda/4\}).$$

Let $\mathcal{A} := \bigcup_i 2Q_{1,i} \cup \bigcup_j 2Q_{2,j}$. Since $\mu(\mathcal{A}) \lesssim \lambda^{-1/2}$, it is enough to consider

$$\mu(\{x \in \mathbb{R}^n \setminus \mathcal{A} : |T_\varepsilon(\beta_1, \beta_2)(x)| > \lambda/4\}).$$

First, we divide $T_\varepsilon(\beta_1, \beta_2)$ into two symmetric parts according to the relative side lengths of the cubes $R_{1,i}$ and $R_{2,j}$. Namely, we have

$$T_\varepsilon(\beta_1, \beta_2) = \sum_i T_\varepsilon\left(\beta_{1,i}, \sum_{\substack{j: \\ \ell(R_{1,i}) \leq \ell(R_{2,j})}} \beta_{2,j}\right) + \sum_j T_\varepsilon\left(\sum_{\substack{i: \\ \ell(R_{1,i}) > \ell(R_{2,j})}} \beta_{1,i}, \beta_{2,j}\right).$$

These two terms are handled symmetrically, so we focus on the first. Define the sets of indices $\mathcal{J}_i := \{j : \ell(R_{1,i}) \leq \ell(R_{2,j})\}$. We have

$$\begin{aligned} & \mu\left(\left\{x \in \mathbb{R}^n \setminus \mathcal{A} : \sum_i \left|T_\varepsilon\left(\beta_{1,i}, \sum_{j \in \mathcal{J}_i} \beta_{2,j}\right)(x)\right| > \lambda/8\right\}\right) \\ (5.14) \quad & \leq \mu\left(\left\{x \in \mathbb{R}^n \setminus \mathcal{A} : \sum_i 1_{(2R_{1,i})^c}(x) \left|T_\varepsilon\left(\beta_{1,i}, \sum_{j \in \mathcal{J}_i} \beta_{2,j}\right)(x)\right| > \lambda/16\right\}\right) \\ & + \mu\left(\left\{x \in \mathbb{R}^n \setminus \mathcal{A} : \sum_i 1_{2R_{1,i}}(x) \left|T_\varepsilon\left(\beta_{1,i}, \sum_{j \in \mathcal{J}_i} \beta_{2,j}\right)(x)\right| > \lambda/16\right\}\right) \\ & =: I + II. \end{aligned}$$

These two cases will be handled separately.

We begin the estimation of I . We have

$$\begin{aligned} \lambda^{1/2} I & \lesssim \int_{\mathbb{R}^n} \left(\sum_{i,j} 1_{(2R_{1,i})^c} 1_{(2R_{2,j})^c} |T_\varepsilon(\beta_{1,i}, \beta_{2,j})| \right)^{1/2} d\mu \\ & + \int_{\mathbb{R}^n \setminus \mathcal{A}} \left(\sum_{i,j} 1_{(2R_{1,i})^c} 1_{2R_{2,j}} |T_\varepsilon(\beta_{1,i}, \beta_{2,j})| \right)^{1/2} d\mu \\ & =: I_a + I_b. \end{aligned}$$

Notice that we dropped the restriction $j \in \mathcal{J}_i$.

To control I_a consider some $x \in (2R_{1,i})^c \cap (2R_{2,j})^c$. Suppose first that $d(x, R_{1,i}) > \varepsilon$. Then, since $R_{1,i} \subset \bar{B}(x, \varepsilon)^c$, we may estimate

$$\begin{aligned} |T_\varepsilon(\beta_{1,i}, \beta_{2,j})(x)| &= \left| \int_{R_{2,j}} \int_{R_{1,i}} [K(x, y, z) - K(x, c_{R_{1,i}}, z)] d\beta_{1,i}(y) d\beta_{2,j}(z) \right| \\ &\lesssim \ell(R_{1,i})^\alpha \frac{|\nu|(Q_{1,i})|\eta|(Q_{2,j})}{(|x - c_{R_{1,i}}| + |x - c_{R_{2,j}}|)^{2m+\alpha}} \\ &\lesssim \ell(R_{1,i})^\alpha \frac{|\nu|(Q_{1,i})}{|x - c_{R_{1,i}}|^{m+\alpha/2}} \frac{|\eta|(Q_{2,j})}{|x - c_{R_{2,j}}|^{m+\alpha/2}}. \end{aligned}$$

We could also use the cancellation in $\beta_{2,j}$, and so it actually holds that

$$|T_\varepsilon(\beta_{1,i}, \beta_{2,j})(x)| \lesssim \ell(R_{1,i})^{\alpha/2} \frac{|\nu|(Q_{1,i})}{|x - c_{R_{1,i}}|^{m+\alpha/2}} \cdot \ell(R_{2,j})^{\alpha/2} \frac{|\eta|(Q_{2,j})}{|x - c_{R_{2,j}}|^{m+\alpha/2}}.$$

This gives that

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\sum_{i,j} 1_{(2R_{1,i})^c} 1_{\{d(\cdot, R_{1,i}) > \varepsilon\}} 1_{(2R_{2,j})^c} |T_\varepsilon(\beta_{1,i}, \beta_{2,j})| \right)^{1/2} d\mu \\ &\lesssim \left(\int_{\mathbb{R}^n} \sum_i 1_{R_{1,i}^c}(x) \frac{\ell(R_{1,i})^{\alpha/2} |\nu|(Q_{1,i})}{|x - c_{R_{1,i}}|^{m+\alpha/2}} d\mu(x) \right)^{1/2} \\ &\quad \times \left(\int_{\mathbb{R}^n} \sum_j 1_{R_{2,j}^c}(x) \frac{\ell(R_{2,j})^{\alpha/2} |\eta|(Q_{2,j})}{|x - c_{R_{2,j}}|^{m+\alpha/2}} d\mu(x) \right)^{1/2} \\ &\lesssim \left(\sum_i |\nu|(Q_{1,i}) \right)^{1/2} \left(\sum_j |\eta|(Q_{2,j}) \right)^{1/2} \lesssim 1. \end{aligned}$$

The size estimate gives for $x \in (2R_{2,j})^c$ that

$$|T_\varepsilon(\beta_{1,i}, \beta_{2,j})(x)| \lesssim |\nu|(Q_{1,i}) \cdot \frac{|\eta|(Q_{2,j})}{(\varepsilon + |x - c_{R_{2,j}}|)^{2m}}.$$

Notice that if $x \in \mathbb{R}^n \setminus 2R_{1,i}$ satisfies $d(x, R_{1,i}) \leq \varepsilon$, then $|x - c_{R_{1,i}}| \lesssim d(x, R_{1,i}) \leq \varepsilon$.

This gives that

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\sum_{i,j} 1_{(2R_{1,i})^c} 1_{\{d(\cdot, R_{1,i}) \leq \varepsilon\}} 1_{(2R_{2,j})^c} |T_\varepsilon(\beta_{1,i}, \beta_{2,j})| \right)^{1/2} d\mu \\ &\lesssim \int_{\mathbb{R}^n} \left(\sum_i 1_{B(c_{R_{1,i}}, C\varepsilon)}(x) |\nu|(Q_{1,i}) \right)^{1/2} \left(\sum_j \frac{|\eta|(Q_{2,j})}{(\varepsilon + |x - c_{R_{2,j}}|)^{2m}} \right)^{1/2} d\mu(x) \\ &\lesssim \left(\sum_i \varepsilon^m |\nu|(Q_{1,i}) \right)^{1/2} \left(\sum_j |\eta|(Q_{2,j}) \varepsilon^{-m} \right)^{1/2} \lesssim 1. \end{aligned}$$

Combining everything, we have shown that $I_a \lesssim 1$.

We turn our attention to the term I_b . We estimate

$$\begin{aligned} I_b &\leq \int_{\mathbb{R}^n} \left(\sum_{i,j} 1_{(2R_{1,i})^c} 1_{2R_{2,j}} |T_\varepsilon(\beta_{1,i}, \varphi_{2,j} d\mu)| \right)^{1/2} d\mu \\ &\quad + \int_{\mathbb{R}^n} \left(\sum_{i,j} 1_{(2R_{1,i})^c} 1_{2R_{2,j} \setminus 2Q_{2,j}} |T_\varepsilon(\beta_{1,i}, w_{2,j} d\eta)| \right)^{1/2} d\mu = I'_b + I''_b. \end{aligned}$$

In the second term we were able to change $1_{2R_{1,i}}$ into $1_{2R_{1,i} \setminus 2Q_{1,i}}$, since the integral in I_b is over $\mathbb{R}^n \setminus \mathcal{A}$. After this change we omitted the restriction of the integral to the complement of \mathcal{A} .

To deal with I'_b we have to once again perform the usual trickery involving the truncation parameter ε . If $x \in (2R_{1,i})^c$ satisfies $d(x, R_{1,i}) > \varepsilon$, then the Hölder estimate in the y -variable yields

$$|T_\varepsilon(\beta_{1,i}, \varphi_{2,j} d\mu)(x)| \lesssim \frac{\ell(R_{1,i})^\alpha |\nu|(Q_{1,i})}{|x - c_{R_{1,i}}|^{m+\alpha}} \|\varphi_{2,j}\|_{L^\infty(\mu)}.$$

This leads to the bound

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\sum_{i,j} 1_{(2R_{1,i})^c} 1_{\{d(\cdot, R_{1,i}) > \varepsilon\}} 1_{2R_{2,j}} |T_\varepsilon(\beta_{1,i}, \varphi_{2,j} d\mu)| \right)^{1/2} d\mu \\ &\lesssim \left(\int_{\mathbb{R}^n} \sum_i 1_{R_{1,i}^c}(x) \frac{\ell(R_{1,i})^\alpha |\nu|(Q_{1,i})}{|x - c_{R_{1,i}}|^{m+\alpha}} d\mu(x) \right)^{1/2} \\ &\quad \times \left(\int_{\mathbb{R}^n} \sum_j \|\varphi_{2,j}\|_{L^\infty(\mu)} 1_{2R_{2,j}}(x) d\mu(x) \right)^{1/2} \lesssim 1. \end{aligned}$$

On the other hand, the size estimate gives

$$|T_\varepsilon(\beta_{1,i}, \varphi_{2,j} d\mu)(x)| \lesssim \frac{|\nu|(Q_{1,i})}{\varepsilon^m} \|\varphi_{2,j}\|_{L^\infty(\mu)},$$

which leads to the bound

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\sum_{i,j} 1_{(2R_{1,i})^c} 1_{\{d(\cdot, R_{1,i}) \leq \varepsilon\}} 1_{2R_{2,j}} |T_\varepsilon(\beta_{1,i}, \varphi_{2,j} d\mu)| \right)^{1/2} d\mu \\ &\lesssim \left(\int_{\mathbb{R}^n} \sum_i 1_{B(c_{R_{1,i}}, C\varepsilon)}(x) \frac{|\nu|(Q_{1,i})}{\varepsilon^m} d\mu(x) \right)^{1/2} \\ &\quad \times \left(\int_{\mathbb{R}^n} \sum_j \|\varphi_{2,j}\|_{L^\infty(\mu)} 1_{2R_{2,j}}(x) d\mu(x) \right)^{1/2} \lesssim 1. \end{aligned}$$

This shows that $I'_b \lesssim 1$.

Let us now control I_b'' . For $x \in (2R_{1,i})^c \cap (2Q_{2,j})^c$ satisfying $d(x, R_{1,i}) > \varepsilon$, we have using the Hölder estimate in the y -variable that

$$|T_\varepsilon(\beta_{1,i}, w_{2,j} d\eta)(x)| \lesssim \frac{\ell(R_{1,i})^\alpha |\nu|(Q_{1,i})}{|x - c_{R_{1,i}}|^{m+\alpha}} \frac{|\eta|(Q_{2,j})}{|x - c_{Q_{2,j}}|^m},$$

which gives the bound

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\sum_{i,j} 1_{(2R_{1,i})^c} 1_{\{d(\cdot, R_{1,i}) > \varepsilon\}} 1_{2R_{2,j} \setminus 2Q_{2,j}} |T_\varepsilon(\beta_{1,i}, w_{2,j} d\eta)| \right)^{1/2} d\mu \\ & \lesssim \left(\int_{\mathbb{R}^n} \sum_i 1_{R_{1,i}^c}(x) \frac{\ell(R_{1,i})^\alpha |\nu|(Q_{1,i})}{|x - c_{R_{1,i}}|^{m+\alpha}} d\mu(x) \right)^{1/2} \\ & \times \left(\int_{\mathbb{R}^n} \sum_j 1_{2R_{2,j} \setminus 2Q_{2,j}} \frac{|\eta|(Q_{2,j})}{|x - c_{Q_{2,j}}|^m} d\mu(x) \right)^{1/2} \lesssim 1. \end{aligned}$$

Here we used (5.12) to estimate the integrals over $2R_{2,j} \setminus 2Q_{2,j}$. For $x \in (2Q_{2,j})^c$ we have using the size estimate that

$$|T_\varepsilon(\beta_{1,i}, w_{2,j} d\eta)(x)| \lesssim \frac{|\nu|(Q_{1,i})}{\varepsilon^m} \frac{|\eta|(Q_{2,j})}{|x - c_{Q_{2,j}}|^m},$$

which leads to the bound

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\sum_{i,j} 1_{(2R_{1,i})^c} 1_{\{d(\cdot, R_{1,i}) \leq \varepsilon\}} 1_{2R_{2,j} \setminus 2Q_{2,j}} |T_\varepsilon(\beta_{1,i}, w_{2,j} d\eta)| \right)^{1/2} d\mu \\ & \lesssim \left(\int_{\mathbb{R}^n} \sum_i 1_{B(c_{R_{1,i}}, C\varepsilon)}(x) \frac{|\nu|(Q_{1,i})}{\varepsilon^m} d\mu(x) \right)^{1/2} \\ & \times \left(\int_{\mathbb{R}^n} \sum_j 1_{2R_{2,j} \setminus 2Q_{2,j}} \frac{|\eta|(Q_{2,j})}{|x - c_{Q_{2,j}}|^m} d\mu(x) \right)^{1/2} \lesssim 1. \end{aligned}$$

Therefore, $I_b'' \lesssim 1$, and so $I_b \lesssim 1$. We have shown that

$$I \lesssim \lambda^{-1/2} (I_a + I_b) \lesssim \lambda^{-1/2}.$$

It remains to show that

$$II = \mu \left(\left\{ x \in \mathbb{R}^n \setminus \mathcal{A} : \sum_i 1_{2R_{1,i}}(x) \left| T_\varepsilon \left(\beta_{1,i}, \sum_{j \in \mathcal{J}_i} \beta_{2,j} \right) (x) \right| > \lambda/16 \right\} \right) \lesssim \lambda^{-1/2}.$$

To reduce things somewhat, notice that

$$\begin{aligned} & \lambda^{1/2} \mu \left(\left\{ x \in \mathbb{R}^n \setminus \mathcal{A} : \sum_i 1_{2R_{1,i}}(x) \left| T_\varepsilon \left(\beta_{1,i}, \sum_{\substack{j \in \mathcal{J}_i \\ 2R_{1,i} \cap 2R_{2,j} = \emptyset}} \beta_{2,j} \right) (x) \right| > \lambda/32 \right\} \right) \\ & \lesssim \int_{\mathbb{R}^n \setminus \mathcal{A}} \left(\sum_{i,j} 1_{2R_{1,i}} 1_{(2R_{2,j})^c} |T_\varepsilon(\beta_{1,i}, \beta_{2,j})| \right)^{1/2} d\mu \lesssim 1, \end{aligned}$$

where we used that the appearing term is similar with I_b from above. Define

$$\tilde{\mathcal{J}}_i := \{j \in \mathcal{J}_i : 2R_{1,i} \cap 2R_{2,j} \neq \emptyset\}.$$

After splitting $\beta_{1,i} = \varphi_{1,i}d\mu + w_{1,i}d\nu$ and $\beta_{2,j} = \varphi_{2,j}d\mu + w_{2,j}d\eta$, what remains to be done is to estimate the following four terms:

$$\begin{aligned} II_a &:= \mu\left(\left\{x \in \mathbb{R}^n \setminus \mathcal{A} : \sum_i 1_{2R_{1,i}}(x) \left|T_{\mu,\varepsilon}\left(\varphi_{1,i}, \sum_{j \in \tilde{\mathcal{J}}_i} \varphi_{2,j}\right)(x)\right| > \lambda/128\right\}\right), \\ II_b &:= \mu\left(\left\{x \in \mathbb{R}^n \setminus \mathcal{A} : \sum_i 1_{2R_{1,i}}(x) \left|T_\varepsilon\left(\varphi_{1,i}d\mu, \sum_{j \in \tilde{\mathcal{J}}_i} w_{2,j}d\eta\right)(x)\right| > \lambda/128\right\}\right), \\ II_c &:= \mu\left(\left\{x \in \mathbb{R}^n \setminus \mathcal{A} : \sum_i 1_{2R_{1,i}}(x) \left|T_\varepsilon\left(w_{1,i}d\nu, \sum_{j \in \tilde{\mathcal{J}}_i} \varphi_{2,j}d\mu\right)(x)\right| > \lambda/128\right\}\right), \\ II_d &:= \mu\left(\left\{x \in \mathbb{R}^n \setminus \mathcal{A} : \sum_i 1_{2R_{1,i}}(x) \left|T_\varepsilon\left(w_{1,i}d\nu, \sum_{j \in \tilde{\mathcal{J}}_i} w_{2,j}d\eta\right)(x)\right| > \lambda/128\right\}\right). \end{aligned}$$

Estimate $II_a \leq II'_a + II''_a$, where

$$II'_a = \mu\left(\left\{x \in \mathbb{R}^n \setminus \mathcal{A} : \sum_i 1_{2R_{1,i}}(x) \left|T_{\mu,\varepsilon}\left(\varphi_{1,i}, 1_{4R_{1,i}} \sum_{j \in \tilde{\mathcal{J}}_i} \varphi_{2,j}\right)(x)\right| > \lambda/256\right\}\right),$$

and II''_a is defined in the obvious way with $1_{4R_{1,i}}$ replaced by $1_{(4R_{1,i})^c}$ inside $T_{\mu,\varepsilon}$. We have using the boundedness of $T_{\mu,\varepsilon}$ that

$$\begin{aligned} II'_a &\lesssim \lambda^{-1} \int \sum_i 1_{2R_{1,i}} \left|T_{\mu,\varepsilon}\left(\varphi_{1,i}, 1_{4R_{1,i}} \sum_{j \in \tilde{\mathcal{J}}_i} \varphi_{2,j}\right)\right| d\mu \\ &\lesssim \lambda^{-1} \sum_i \mu(R_{1,i})^{1-1/r} \|\varphi_{1,i}\|_{L^\infty(\mu)} \mu(R_{1,i})^{1/p} \left\| \sum_j |\varphi_{2,j}| \right\|_{L^\infty(\mu)} \mu(R_{1,i})^{1/q} \\ &\lesssim \lambda^{-1/2} \sum_i \mu(R_{1,i}) \|\varphi_{1,i}\|_{L^\infty(\mu)} \lesssim \lambda^{-1/2}. \end{aligned}$$

Notice that we used that $\sum_j |\varphi_{2,j}| \lesssim \lambda^{1/2}$. Using the size estimate we see that for $x \in 2R_{1,i}$ it holds that

$$\begin{aligned} \left|T_{\mu,\varepsilon}\left(\varphi_{1,i}, 1_{(4R_{1,i})^c} \sum_{j \in \tilde{\mathcal{J}}_i} \varphi_{2,j}\right)\right| &\lesssim \|\varphi_{1,i}\|_{L^\infty(\mu)} \left\| \sum_j |\varphi_{2,j}| \right\|_{L^\infty(\mu)} \int_{R_{1,i}^c} \int_{R_{1,i}} \frac{d\mu(y) d\mu(z)}{|z - c_{R_{1,i}}|^{2m}} \\ &\lesssim \lambda^{1/2} \|\varphi_{1,i}\|_{L^\infty(\mu)} \frac{\mu(R_{1,i})}{\ell(R_{1,i})^m} \lesssim \lambda^{1/2} \|\varphi_{1,i}\|_{L^\infty(\mu)}. \end{aligned}$$

Therefore, we have

$$II''_a \lesssim \lambda^{-1} \int \sum_i 1_{2R_{1,i}} \lambda^{1/2} \|\varphi_{1,i}\|_{L^\infty(\mu)} d\mu \lesssim \lambda^{-1/2},$$

and this completes the proof of the fact that $II_a \lesssim \lambda^{-1/2}$.

For II_b , notice that the size estimate gives for $x \in \mathbb{R}^n \setminus \mathcal{A} \subset (2Q_{2,j})^c$ that

$$|T_\varepsilon(\varphi_{1,i} d\mu, w_{2,j} d\eta)(x)| \lesssim \frac{\|\varphi_{1,i}\|_{L^\infty(\mu)} |\eta|(Q_{2,j})}{|x - c_{Q_{2,j}}|^m}.$$

Notice also that if $j \in \tilde{\mathcal{J}}_i$, then $2R_{1,i} \subset 6R_{2,j}$. Therefore, we get

$$\lambda^{1/2} II_b \lesssim \int_{\mathbb{R}^n} \left(\sum_{i,j} 1_{2R_{1,i}}(x) 1_{6R_{2,j} \setminus Q_{2,j}}(x) \frac{\|\varphi_{1,i}\|_{L^\infty(\mu)} |\eta|(Q_{2,j})}{|x - c_{Q_{2,j}}|^m} \right)^{1/2} d\mu(x).$$

From here the estimate is concluded as before, using first Hölder's inequality in $L^2(\mu)$, and then estimating the resulting two integrals with the help of equations (5.8) and (5.12). This shows that $II_b \lesssim \lambda^{-1/2}$. The estimate $II_c \lesssim \lambda^{-1/2}$ is concluded with essentially same arguments. Regarding the term II_d , we have

$$|T_\varepsilon(w_{1,i} d\nu, w_{2,j} d\eta)(x)| \lesssim \frac{|\nu|(Q_{1,i}) |\eta|(Q_{2,j})}{|x - c_{Q_{1,i}}|^m |x - c_{Q_{2,j}}|^m}, \quad x \in (2Q_{1,i})^c \cap (2Q_{2,j})^c.$$

This allows to estimate II_d with similar steps as we used with II_b .

This finally almost concludes the proof. It remains to note that it is straightforward to drop the assumption that ν and η have compact support. The argument goes quite similarly as in the linear case (see e.g. [24]). \square

Next, we prove a version of Cotlar's inequality.

5.15. Proposition. *Let μ be a measure of order m on \mathbb{R}^n and T be a bilinear m -dimensional SIO. Let $\delta > 0$ and suppose that*

$$\|T_\delta\|_{M(\mathbb{R}^n) \times M(\mathbb{R}^n) \rightarrow L^{1/2,\infty}(\mu)} \lesssim 1.$$

Then for all $\nu_1, \nu_2 \in M(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we have uniformly on $\varepsilon > \delta$ that

$$|T_\varepsilon(\nu_1, \nu_2)(x)| \lesssim N_{\mu,1/4}(T_\delta(\nu_1, \nu_2))(x) + M_\mu \nu_1(x) M_\mu \nu_2(x).$$

Proof. Fix $\nu_1, \nu_2 \in M(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $\varepsilon_0 > 0$. We will estimate $|T_{\varepsilon_0}(\nu_1, \nu_2)(x)|$. For convenience, we will throughout the proof denote restrictions of measures, like $\nu|_A$, with $1_A \nu$. Choose the smallest k so that $B(x, 5^k \varepsilon_0)$ is $(5, 5^{m+1})$ doubling with respect to μ . Set $\varepsilon = 5^k \varepsilon_0$.

We will begin by controlling $|T_{\varepsilon_0}(\nu_1, \nu_2)(x) - T_\varepsilon(\nu_1, \nu_2)(x)|$ – which is a standard argument for experts. Notice that

$$\begin{aligned} & |T_{\varepsilon_0}(\nu_1, \nu_2)(x) - T_\varepsilon(\nu_1, \nu_2)(x)| \\ & \lesssim \iint_{\varepsilon_0 < \max(|x-y|, |x-z|) \leq \varepsilon} \frac{d|\nu_1|(y) d|\nu_2|(z)}{(|x-y| + |x-z|)^{2m}} \\ & \lesssim M_\mu \nu_2(x) \int_{\varepsilon_0 < |x-y| \leq \varepsilon} \frac{d|\nu_1|(y)}{|x-y|^m} + M_\mu \nu_1(x) \int_{\varepsilon_0 < |x-z| \leq \varepsilon} \frac{d|\nu_2|(z)}{|x-z|^m}. \end{aligned}$$

These terms are completely symmetric, so it suffices to deal with the first. Notice that

$$\int_{|x-y|=\varepsilon} \frac{d|\nu_1|(y)}{|x-y|^m} \leq \frac{|\nu_1|(B(x, 2\varepsilon))}{\varepsilon^m} \lesssim M_\mu \nu_1(x).$$

We bound

$$\begin{aligned} \int_{\varepsilon_0 < |x-y| < \varepsilon} \frac{d|\nu_1|(y)}{|x-y|^m} &\leq \sum_{j=0}^{k-1} \int_{5^j \varepsilon_0 \leq |x-y| < 5^{j+1} \varepsilon_0} \frac{d|\nu_1|(y)}{|x-y|^m} \\ &\leq \sum_{j=0}^{k-1} (5^j \varepsilon_0)^{-m} |\nu_1|(B(x, 5^{j+1} \varepsilon_0)) \\ &\leq M_\mu \nu_1(x) \sum_{j=0}^{k-1} (5^j \varepsilon_0)^{-m} \mu(B(x, 5^{j+1} \varepsilon_0)). \end{aligned}$$

Since

$$\mu(B(x, 5^{j+1} \varepsilon_0)) \leq (5^{-m-1})^{k-j-1} \mu(B(x, \varepsilon)) \lesssim (5^{-m-1})^{k-j} \varepsilon^m = (5^{-m-1})^{k-j} (2^k \varepsilon_0)^m,$$

it follows that

$$\sum_{j=0}^{k-1} (5^j \varepsilon_0)^{-m} \mu(B(x, 5^{j+1} \varepsilon_0)) \lesssim \sum_{j=0}^{\infty} \left(\frac{1}{5}\right)^j \lesssim 1.$$

We have shown that

$$|T_{\varepsilon_0}(\nu_1, \nu_2)(x)| \lesssim |T_\varepsilon(\nu_1, \nu_2)(x)| + M_\mu \nu_1(x) M_\mu \nu_2(x),$$

and so are reduced to bounding $|T_\varepsilon(\nu_1, \nu_2)(x)|$.

For a fixed $w \in B(x, \varepsilon)$ write

$$\begin{aligned} T_\varepsilon(\nu_1, \nu_2)(x) &= T_\varepsilon(\nu_1, \nu_2)(x) - T_\delta(1_{B(x, 2\varepsilon)^c} \nu_1, \nu_2)(w) \\ &\quad + T_\delta(\nu_1, \nu_2)(w) - T_\delta(1_{B(x, 2\varepsilon)} \nu_1, \nu_2)(w). \end{aligned}$$

Now for every $w \in B(x, \varepsilon)$ we have $B(w, \delta) \subset B(x, 2\varepsilon)$ so that we can dominate

$$|T_\varepsilon(\nu_1, \nu_2)(x) - T_\delta(1_{B(x, 2\varepsilon)^c} \nu_1, \nu_2)(w)|$$

with the sum of

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{B(x, 2\varepsilon)^c} |K(x, y, z) - K(w, y, z)| d|\nu_1|(y) d|\nu_2|(z) \\ &\lesssim \int_{B(x, 2\varepsilon)^c} \int_{\mathbb{R}^n} \frac{\varepsilon^\alpha}{(|x-y| + |x-z|)^{2m+\alpha}} d|\nu_2|(z) d|\nu_1|(y) \\ &\lesssim M_m \nu_2(x) \cdot \varepsilon^\alpha \int_{B(x, \varepsilon)^c} \frac{d|\nu_1|(y)}{|x-y|^{m+\alpha}} \lesssim M_m \nu_1(x) M_m \nu_2(x) \lesssim M_\mu \nu_1(x) M_\mu \nu_2(x). \end{aligned}$$

and

$$\begin{aligned} \iint_{\substack{\max(|x-y|, |x-z|) > \varepsilon \\ y \in B(x, 2\varepsilon)}} |K(x, y, z)| d|\nu_1|(y) d|\nu_2|(z) &\lesssim \int_{B(x, 2\varepsilon)} \int \frac{d|\nu_2|(z)}{(\varepsilon + |x - z|)^{2m}} d|\nu_1|(y) \\ &\lesssim \frac{|\nu_1|(B(x, 2\varepsilon))}{\varepsilon^m} M_\mu \nu_2(x) \\ &\lesssim M_\mu \nu_1(x) M_\mu \nu_2(x). \end{aligned}$$

The above shows that for all $w \in B(x, \varepsilon)$ we have

$$|T_\varepsilon(\nu_1, \nu_2)(x)| \lesssim M_\mu \nu_1(x) M_\mu \nu_2(x) + |T_\delta(\nu_1, \nu_2)(w)| + |T_\delta(1_{B(x, 2\varepsilon)} \nu_1, \nu_2)(w)|.$$

It follows from this by raising to the power $1/4$, averaging over $w \in B(x, \varepsilon)$ and raising to power 4 that

$$|T_\varepsilon(\nu_1, \nu_2)(x)| \lesssim M_\mu \nu_1(x) M_\mu \nu_2(x) + I + II,$$

where

$$I := \left(\frac{1}{\mu(B(x, \varepsilon))} \int_{B(x, \varepsilon)} |T_\delta(\nu_1, \nu_2)(w)|^{1/4} d\mu(w) \right)^4$$

and

$$II := \left(\frac{1}{\mu(B(x, \varepsilon))} \int_{B(x, \varepsilon)} |T_\delta(1_{B(x, 2\varepsilon)} \nu_1, \nu_2)(w)|^{1/4} d\mu(w) \right)^4.$$

Since $\mu(B(x, 5\varepsilon)) \leq 5^{m+1} \mu(B(x, \varepsilon))$ we have

$$I \lesssim N_{\mu, 1/4}(T_\delta(\nu_1, \nu_2))(x).$$

It only remains to estimate the term II , which we begin by estimating

$$\begin{aligned} II &\lesssim \left(\frac{1}{\mu(B(x, \varepsilon))} \int_{B(x, \varepsilon)} |T_\delta(1_{B(x, 2\varepsilon)} \nu_1, 1_{B(x, 2\varepsilon)^c} \nu_2)(w)|^{1/4} d\mu(w) \right)^4 \\ &\quad + \left(\frac{1}{\mu(B(x, \varepsilon))} \int_{B(x, \varepsilon)} |T_\delta(1_{B(x, 2\varepsilon)} \nu_1, 1_{B(x, 2\varepsilon)} \nu_2)(w)|^{1/4} d\mu(w) \right)^4 = II' + II''. \end{aligned}$$

Notice that for all $w \in B(x, \varepsilon)$ we have

$$\begin{aligned} |T_\delta(1_{B(x, 2\varepsilon)} \nu_1, 1_{B(x, 2\varepsilon)^c} \nu_2)(w)| &\lesssim \int_{B(x, 2\varepsilon)} \int_{B(x, 2\varepsilon)^c} \frac{d|\nu_2|(z)}{|w - z|^{2m}} d|\nu_1|(y) \\ &\lesssim |\nu_1|(B(x, 2\varepsilon)) \int_{B(x, \varepsilon)^c} \frac{d|\nu_2|(z)}{|z - x|^{2m}} \\ &\lesssim M_m \nu_2(x) \cdot \frac{|\nu_1|(B(x, 2\varepsilon))}{\varepsilon^m} \lesssim M_\mu \nu_1(x) M_\mu \nu_2(x) \end{aligned}$$

so that

$$II' \lesssim M_\mu \nu_1(x) M_\mu \nu_2(x).$$

We are left with II'' , which we will handle using the assumption

$$\|T_\delta\|_{M(\mathbb{R}^n) \times M(\mathbb{R}^n) \rightarrow L^{1/2, \infty}(\mu)} \lesssim 1,$$

the doubling property of the ball $B(x, \varepsilon)$ and some Kolmogorov type arguments. We have

$$\begin{aligned}
& \int_{B(x, \varepsilon)} |T_\delta(1_{B(x, 2\varepsilon)}\nu_1, 1_{B(x, 2\varepsilon)}\nu_2)(w)|^{1/4} d\mu(w) \\
&= \frac{1}{4} \int_0^\infty \lambda^{-3/4} \mu(\{w \in B(x, \varepsilon) : |T_\delta(1_{B(x, 2\varepsilon)}\nu_1, 1_{B(x, 2\varepsilon)}\nu_2)(w)| > \lambda\}) d\lambda \\
&\lesssim \int_0^A \lambda^{-3/4} d\lambda \cdot \mu(B(x, \varepsilon)) + \int_A^\infty \lambda^{-5/4} d\lambda \cdot |\nu_1|(B(x, 2\varepsilon))^{1/2} |\nu_2|(B(x, 2\varepsilon))^{1/2} \\
&\lesssim A^{1/4} \mu(B(x, \varepsilon)) + A^{-1/4} |\nu_1|(B(x, 2\varepsilon))^{1/2} |\nu_2|(B(x, 2\varepsilon))^{1/2} \\
&\lesssim |\nu_1|(B(x, 2\varepsilon))^{1/4} |\nu_2|(B(x, 2\varepsilon))^{1/4} \mu(B(x, \varepsilon))^{1/2},
\end{aligned}$$

where we used the choice

$$A = \frac{|\nu_1|(B(x, 2\varepsilon)) |\nu_2|(B(x, 2\varepsilon))}{\mu(B(x, \varepsilon))^2}.$$

This gives

$$\begin{aligned}
II'' &\lesssim \frac{|\nu_1|(B(x, 2\varepsilon))}{\mu(B(x, \varepsilon))} \frac{|\nu_2|(B(x, 2\varepsilon))}{\mu(B(x, \varepsilon))} \\
&\lesssim \frac{|\nu_1|(B(x, 2\varepsilon))}{\mu(B(x, 2\varepsilon))} \frac{|\nu_2|(B(x, 2\varepsilon))}{\mu(B(x, 2\varepsilon))} \lesssim M_\mu \nu_1(x) M_\mu \nu_2(x),
\end{aligned}$$

and completes the proof. \square

Finally, we can get the weak type bound for $T_\#$.

5.16. Proposition. *Let μ be a measure of order m on \mathbb{R}^n and T be a bilinear m -dimensional SIO. Let $1 < r, p, q < \infty$ be so that $1/p + 1/q = 1/r$, and suppose we have uniformly on $\varepsilon > 0$ that*

$$\|T_{\mu, \varepsilon}\|_{L^p(\mu) \times L^q(\mu) \rightarrow L^r(\mu)} \lesssim 1.$$

Then we have

$$\|T_\#\|_{M(\mathbb{R}^n) \times M(\mathbb{R}^n) \rightarrow L^{1/2, \infty}(\mu)} \lesssim 1.$$

Proof. Fix $\nu_1, \nu_2 \in M(\mathbb{R}^n)$. It suffices to prove that

$$\sup_{\delta > 0} \sup_{\lambda > 0} \lambda \mu(\{x \in \mathbb{R}^n : T_{\#, \delta}(\nu_1, \nu_2)(x) > \lambda\})^2 \lesssim \|\nu_1\| \|\nu_2\|.$$

Fix $\delta > 0$. We know by Proposition 5.9 that

$$\|T_\delta\|_{M(\mathbb{R}^n) \times M(\mathbb{R}^n) \rightarrow L^{1/2, \infty}(\mu)} \lesssim 1.$$

In particular, we have by Proposition 5.15 that

$$T_{\#, \delta}(\nu_1, \nu_2)(x) \lesssim N_{\mu, 1/4}(T_\delta(\nu_1, \nu_2))(x) + M_\mu \nu_1(x) M_\mu \nu_2(x), \quad x \in \mathbb{R}^n.$$

Recall that

$$\sup_{\lambda > 0} \lambda \mu(\{x \in \mathbb{R}^n : M_\mu \nu_1(x) M_\mu \nu_2(x) > \lambda\})^2 \lesssim \|\nu_1\| \|\nu_2\|.$$

This is probably easiest to see by using the facts that

$$\|fg\|_{L^{1/2,\infty}(\mu)} \lesssim \|f\|_{L^{1,\infty}(\mu)} \|g\|_{L^{1,\infty}(\mu)}$$

and $\|M_\mu\|_{M(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mu)} \lesssim 1$.

Thus, it suffices to prove that

$$\sup_{\lambda>0} \lambda \mu(\{x \in \mathbb{R}^n : N_{\mu,1/4}(T_\delta(\nu_1, \nu_2))(x) > \lambda\})^2 \lesssim \|\nu_1\| \|\nu_2\|.$$

We will next use the easy fact that

$$\sup_{\lambda>0} \lambda \mu(\{x \in \mathbb{R}^n : N_\mu \nu(x) > \lambda\}) \leq |\nu|(\{x \in \mathbb{R}^n : N_\mu \nu(x) > \lambda\}), \quad \nu \in M(\mathbb{R}^n).$$

This sharper form of the weak $(1, 1)$ inequality is the only reason why the non-homogeneous, non-centered maximal function N_μ is important in the Cotlar's inequality. Fix $\lambda > 0$ and set

$$\begin{aligned} H &= \{x \in \mathbb{R}^n : N_{\mu,1/4}(T_\delta(\nu_1, \nu_2))(x) > \lambda\} \\ &= \{x \in \mathbb{R}^n : N_\mu(|T_\delta(\nu_1, \nu_2)|^{1/4})(x) > \lambda^{1/4}\}. \end{aligned}$$

We now have

$$\mu(H) \leq \frac{1}{\lambda^{1/4}} \int_H |T_\delta(\nu_1, \nu_2)|^{1/4} d\mu,$$

where

$$\int_H |T_\delta(\nu_1, \nu_2)|^{1/4} d\mu \lesssim \|\nu_1\|^{1/4} \|\nu_2\|^{1/4} \mu(H)^{1/2}$$

using again that $\|T_\delta\|_{M(\mathbb{R}^n) \times M(\mathbb{R}^n) \rightarrow L^{1/2,\infty}(\mu)} \lesssim 1$ and the Kolmogorov type argument from the proof of Proposition 5.15. Therefore, we have

$$\mu(H) \lesssim \left(\frac{\|\nu_1\| \|\nu_2\|}{\lambda} \right)^{1/2},$$

and we are done. \square

6. BILINEAR GOOD LAMBDA METHOD

In this section we aim to prove Theorem 6.3 – a certain very useful good lambda type result. The proof of this in the bilinear setting turns out not to be very different from the linear setting presented in Theorem 2.22 in [24]. For the convenience of the reader we give most of the details here.

Before proving the good lambda, we need to recall the following version of Whitney covering especially useful for non-doubling measures. This is originally from [24], but the version with small boundary cubes as here appears in [18].

6.1. Lemma. *If $\Omega \subset \mathbb{R}^n$ is open, $\Omega \neq \mathbb{R}^n$, then Ω can be decomposed as*

$$\Omega = \bigcup_{i \in I} Q_i,$$

where Q_i , $i \in I$, are closed dyadic cubes with disjoint interiors such that for some constants $R > 20$ and $D_0 \geq 1$, depending only on n , the following holds:

- (i) $10Q_i \subset \Omega$ for each $i \in I$.
- (ii) $RQ_i \cap \Omega^c \neq \emptyset$ for each $i \in I$.
- (iii) For each cube Q_i , there are at most D_0 cubes Q_j such that $10Q_i \cap 10Q_j \neq \emptyset$.
Further, for such cubes Q_i, Q_j , we have $\ell(Q_i) \sim \ell(Q_j)$.

Moreover, if t is a large enough dimensional constant, μ is a positive Radon measure on \mathbb{R}^n with $\mu(\Omega) < \infty$, there is a family of cubes $\{\tilde{Q}_j\}_{j \in S}$, with $S \subset I$, so that $Q_j \subset \tilde{Q}_j \subset 1.1Q_j$, satisfying the following:

- (a) Each cube \tilde{Q}_j , $j \in S$, is $(9, 2D_0)$ -doubling and has t -small boundary.
- (b) The cubes \tilde{Q}_j , $j \in S$, are pairwise disjoint.
- (c)

$$(6.2) \quad \mu\left(\bigcup_{j \in S} \tilde{Q}_j\right) \geq \frac{1}{8D_0} \mu(\Omega).$$

We are ready for the main result of this section.

6.3. Theorem. Let μ be a measure of order m in \mathbb{R}^n . Let $\beta > 0$ and $t > 0$ be big enough numbers, depending only on the dimension n , and assume $\theta \in (0, 1)$. Suppose for each $(2, \beta)$ -doubling cube Q with t -small boundary there exists a subset $G_Q \subset Q$ such that $\mu(G_Q) \geq \theta\mu(Q)$ and $T_{\mu, \#}: M(\mathbb{R}^n) \times M(\mathbb{R}^n) \rightarrow L^{1/2, \infty}(\mu|_{G_Q})$ is bounded with a uniform constant independent of Q . Then we have that $T_{\mu, \#}: L^p(\mu) \times L^q(\mu) \rightarrow L^r(\mu)$ boundedly for all $1 < p, q < \infty$ and $1/2 < r < \infty$ satisfying $1/r = 1/p + 1/q$ with a constant depending on r, p, q and the preceding constants.

Proof. Let us fix two functions $f, g \in L^1(\mu)$ with compact support. For $\lambda > 0$ let

$$\Omega_\lambda = \{T_{\mu, \#}(f, g) > \lambda\}.$$

Because the functions f and g have compact support, Ω_λ is a bounded set, and accordingly of finite μ -measure. Let us also check the fact that it is open, and for this suppose $x \in \Omega_\lambda$. Thus, there exists $\rho_0 > 0$ so that $T_{\mu, \rho_0}(f, g)(x) > \lambda$. Because the measure $\mu \times \mu$ is locally finite, we can find a slightly larger $\rho > \rho_0$ so that $T_{\mu, \rho}(f, g)(x) > \lambda$ and

$$\mu \times \mu(\{(y, z): \max(|x - y|, |x - z|) = \rho\}) = 0.$$

Then it follows from the dominated convergence theorem that

$$T_{\mu, \rho}(f, g)(x') \rightarrow T_{\mu, \rho}(f, g)(x), \quad x' \rightarrow x,$$

which shows that $T_{\mu, \#}(f, g)(x') > \lambda$ if $|x' - x|$ is small enough. Hence Ω_λ is an open set.

Now we can use Lemma 6.1 to write

$$\Omega_\lambda = \bigcup_{i \in I} Q_i,$$

and also to extract the collection $\{\tilde{Q}_j\}_{j \in S}$, where $S \subset I$, so that all the properties of the lemma hold. For $j \in S$ let us write $P_j = \tilde{Q}_j$. The cubes P_j have

t -small boundary and are $(9, 2D_0)$ -doubling, in particular $(2, 2D_0)$ -doubling. So assuming that the parameter β from the assumptions is larger than $2D_0$, we have by assumption that there exists $G_{P_j} \subset P_j$ so that $\mu(G_{P_j}) \geq \theta\mu(P_j)$ and $T_{\sharp}: M(\mathbb{R}^n) \times M(\mathbb{R}^n) \rightarrow L^{1/2, \infty}(\mu|_{G_{P_j}})$ boundedly with a constant A that is uniform in $j \in S$. For $j \in S$ denote $G_j = G_{P_j}$.

The idea is to prove using the previous cubes that given $\varepsilon, \lambda > 0$ there exists $\delta = \delta(\varepsilon, \theta, A) = \delta(\varepsilon) > 0$ (θ and A are fixed constants from the assumptions) so that

$$(6.4) \quad \mu(\{x: T_{\mu, \sharp}(f, g)(x) > (1+\varepsilon)\lambda, M_{\mu}^{\mathcal{Q}}f(x)M_{\mu}^{\mathcal{Q}}g(x) \leq \delta\lambda\}) \leq \left(1 - \frac{\theta}{16D_0}\right)\mu(\Omega_{\lambda}).$$

This is enough to conclude the whole proof by standard considerations, but we shall quickly recall the necessary steps later.

By exploiting the fact that now

$$\mu\left(\Omega_{\lambda} \setminus \bigcup_{j \in S} P_j\right) + \sum_{j \in S} \mu(P_j \setminus G_j) \leq \left(1 - \frac{\theta}{8D_0}\right)\mu(\Omega_{\lambda}),$$

we are reduced to proving

$$(6.5) \quad \sum_{j \in S} \mu(\{x \in G_j: T_{\mu, \sharp}(f, g)(x) > (1+\varepsilon)\lambda, M_{\mu}^{\mathcal{Q}}f(x)M_{\mu}^{\mathcal{Q}}g(x) \leq \delta\lambda\}) \leq \frac{\theta}{16D_0}\mu(\Omega_{\lambda})$$

if $\delta = \delta(\varepsilon) > 0$ is small enough. We will do this by showing for every fixed j , if $x \in P_j$ is such that

$$T_{\mu, \sharp}(f, g)(x) > (1+\varepsilon)\lambda \quad \text{and} \quad M_{\mu}^{\mathcal{Q}}f(x)M_{\mu}^{\mathcal{Q}}g(x) \leq \delta\lambda$$

and δ is small enough, then

$$(6.6) \quad T_{\mu, \sharp}(f1_{2P_j}, g1_{2P_j})(x) > \frac{\varepsilon}{2}\lambda.$$

This implies (6.5) (for $\delta(\varepsilon)$ small enough) by the fact that $T_{\sharp}: M(\mathbb{R}^n) \times M(\mathbb{R}^n) \rightarrow L^{1/2, \infty}(\mu|_{G_{P_j}})$. This calculation is done almost exactly as in the linear case, but let us quickly check it. So we assume the above pointwise bound. For a fixed $j \in S$ notice that now

$$\begin{aligned} & \mu(\{x \in G_j: T_{\mu, \sharp}(f, g)(x) > (1+\varepsilon)\lambda, M_{\mu}^{\mathcal{Q}}f(x)M_{\mu}^{\mathcal{Q}}g(x) \leq \delta\lambda\}) \\ & \leq \mu(\{x \in G_j: T_{\mu, \sharp}(f1_{2P_j}, g1_{2P_j})(x) > \varepsilon\lambda/2\}) \\ & \leq \left(\frac{2A}{\varepsilon\lambda}\right)^{1/2} \left(\int_{2P_j} |f| d\mu\right)^{1/2} \left(\int_{2P_j} |g| d\mu\right)^{1/2}. \end{aligned}$$

We can assume that there exists $x_0 \in P_j$ such that $M_\mu^\mathcal{Q}f(x_0)M_\mu^\mathcal{Q}g(x_0) \leq \delta\lambda$, and estimate

$$\begin{aligned} & \left(\int_{2P_j} |f| d\mu \right)^{1/2} \left(\int_{2P_j} |g| d\mu \right)^{1/2} \\ & \leq (\mu(10Q_j)M_\mu^\mathcal{Q}f(x_0))^{1/2} (\mu(10Q_j)M_\mu^\mathcal{Q}g(x_0))^{1/2} \leq (\delta\lambda)^{1/2} \mu(10Q_j). \end{aligned}$$

So we have

$$\begin{aligned} & \sum_{j \in S} \mu(\{x \in G_j : T_{\mu, \#}(f, g)(x) > (1 + \varepsilon)\lambda, M_\mu^\mathcal{Q}f(x)M_\mu^\mathcal{Q}g(x) \leq \delta\lambda\}) \\ & \leq \left(\frac{2A\delta}{\varepsilon} \right)^{1/2} \sum_{j \in I} \mu(10Q_j) \leq D_0 \left(\frac{2A\delta}{\varepsilon} \right)^{1/2} \mu(\Omega_\lambda) \leq \frac{\theta}{16D_0} \mu(\Omega_\lambda) \end{aligned}$$

for δ small enough.

It only remains to prove the pointwise lower bound. So suppose $x \in P_j$ is such that

$$T_{\mu, \#}(f, g)(x) > (1 + \varepsilon)\lambda \quad \text{and} \quad M_\mu^\mathcal{Q}f(x)M_\mu^\mathcal{Q}g(x) \leq \delta\lambda.$$

Using the \tilde{T} notation there holds

$$T_{\mu, \#}(f, g)(x) \leq T_{\mu, \#}(1_{2P_j}f, 1_{2P_j}g)(x) + \tilde{T}_{\mu, \#}(1_{(2P_j \times 2P_j)^c}f \otimes g)(x).$$

Therefore, (6.6) follows once we show that the latter term on the right is at most $(1 + \varepsilon/2)\lambda$. To do this, we fix an arbitrary $\rho_0 > 0$ and show that $\tilde{T}_{\mu, \rho_0}(1_{(2P_j \times 2P_j)^c}f \otimes g)(x) \leq (1 + \varepsilon/2)\lambda$. Notice first that

$$\begin{aligned} & \iint_{2 \operatorname{diam}(RP_j) \geq \max(|x-y|, |x-z|)} \frac{1_{(2P_j \times 2P_j)^c}(y, z) |f(y)g(z)|}{(|x-y| + |x-z|)^{2m}} d\mu(y) d\mu(z) \\ & \lesssim \iint_{\substack{|x-y| \leq 2 \operatorname{diam}(RP_j) \\ |x-z| \leq 2 \operatorname{diam}(RP_j)}} \frac{|f(y)g(z)|}{\ell(P_j)^{2m}} d\mu(y) d\mu(z) \\ & \lesssim M_\mu^\mathcal{Q}f(x)M_\mu^\mathcal{Q}g(x) \leq \delta\lambda. \end{aligned}$$

Thus, it is enough to define $\rho := \max(\rho_0, 2 \operatorname{diam}(RP_j))$ and consider

$$\tilde{T}_{\mu, \rho}(1_{(2P_j \times 2P_j)^c}f \otimes g)(x) = \tilde{T}_{\mu, \rho}(f \otimes g)(x) = T_{\mu, \rho}(f, g)(x),$$

where the first equality holds because

$$\{(y, z) : \max(|x-y|, |x-z|) > \rho\} \subset (2P_j \times 2P_j)^c.$$

By the properties of the Whitney cubes there exists a point $x' \in RP_j \cap \Omega_\lambda^c$, and we can estimate

$$\begin{aligned} |T_{\mu, \rho}(f, g)(x)| & \leq |T_{\mu, \rho}(f, g)(x) - T_{\mu, \rho}(f, g)(x')| + |T_{\mu, \rho}(f, g)(x')| \\ & \leq |T_{\mu, \rho}(f, g)(x) - T_{\mu, \rho}(f, g)(x')| + \lambda. \end{aligned}$$

Thus, it suffices to estimate the difference, which can be dominated with

$$(6.7) \quad \left| \iint_{\max(|x-y|, |x-z|) > \rho} (K(x, y, z) - K(x', y, z)) f(y) g(z) d\mu(y) d\mu(z) \right| \\ + \left| \iint_{\max(|x-y|, |x-z|) > \rho} K(x', y, z) f(y) g(z) d\mu(y) d\mu(z) - T_{\mu, \rho}(f, g)(x') \right|.$$

Applying kernel estimates, the first term in (6.7) can be dominated with

$$\iint_{\max(|x-y|, |x-z|) > \rho} \frac{|x - x'|^\alpha |f(y)g(z)|}{(|x - y| + |x - z|)^{2m+\alpha}} d\mu(y) d\mu(z) \lesssim M_\mu^\mathcal{Q} f(x) M_\mu^\mathcal{Q} g(x) \leq \delta \lambda.$$

Notice that the symmetric difference

$$\{(y, z) : \max(|x - y|, |x - z|) > \rho\} \Delta \{(y, z) : \max(|x' - y|, |x' - z|) > \rho\}$$

is contained in the set

$$\{(y, z) : \max(|x - y|, |x - z|) \sim \max(|x' - y|, |x' - z|) \sim \rho\}.$$

Thus, the second term in (6.7) can be dominated with

$$\iint_{\max(|x-y|, |x-z|) \sim \rho} \frac{|f(y)g(z)|}{(|x - y| + |x - z|)^{2m}} d\mu(y) d\mu(z) \lesssim M_\mu^\mathcal{Q} f(x) M_\mu^\mathcal{Q} g(x) \leq \delta \lambda.$$

Combining the above arguments we have shown that there exists a constant C such that

$$\tilde{T}_{\mu, \rho_0}(1_{(2P_j \times 2P_j)^c} f \otimes g)(x) \leq C\delta\lambda + \lambda.$$

Hence, if $\delta(\varepsilon)$ is chosen to be small enough, it is seen that $C\delta\lambda + \lambda \leq (1 + \varepsilon/2)\lambda$, and accordingly (6.6) is satisfied.

We have shown (6.4). The claim follows from this by standard arguments, but requires a moderate amount of approximation. For convenience, we outline these details now. Let $1 < p, q < \infty$ and $1/2 < r < \infty$ satisfy $1/r = 1/p + 1/q$. Since we do not know that $\|T_{\mu, \#}(f, g)\|_{L^r(\mu)} < \infty$, we first do the following. Define $h_k = \inf(k, T_{\mu, \#}(f, g))$, $k \geq 1$. Suppose $R > 0$ is such that $B(0, R)$ contains the supports of f and g . Then for $x \in B(0, 2R)^c$ there holds that

$$h_k(x) \lesssim \frac{\|f\|_{L^1(\mu)} \|g\|_{L^1(\mu)}}{|x|^{2m}}.$$

Notice that $2mr > m$ so that $\int_{B(0, 2R)^c} |x|^{-2mr} d\mu(x) < \infty$. It follows that $\|h_k\|_{L^r(\mu)} < \infty$. Moreover, the good lambda inequality (6.4) is also true with $T_{\mu, \#}(f, g)$ replaced by h_k everywhere (this follows from (6.4) directly using the definition of h_k). Using this good lambda inequality, the fact that $\|M_\mu^\mathcal{Q} f M_\mu^\mathcal{Q} g\|_{L^r(\mu)} \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}$ and $\|h_k\|_{L^r(\mu)} < \infty$, we easily see using the distributional formula for the $L^r(\mu)$ norm that

$$\|h_k\|_{L^r(\mu)} \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

Letting $k \rightarrow \infty$ we get

$$(6.8) \quad \|T_{\mu,\#}(f, g)\|_{L^r(\mu)} \lesssim \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}.$$

Recall that f and g were $L^1(\mu)$ functions with compact supports. Let us now extend this to all functions $f \in L^p(\mu)$ and $g \in L^q(\mu)$. To this end, choose some arbitrary real numbers $M, \rho > 0$. Using (1.1) we see that there exists a constant $C(M, \rho)$ so that

$$(6.9) \quad \|1_{B(0,M)} T_{\mu,\#,\rho}(f, g)\|_{L^r(\mu)} \leq C(M, \rho) \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}, \quad f \in L^p(\mu), g \in L^q(\mu).$$

Approximating with $L^1(\mu)$ functions with compact supports and using (6.8) we see that the above holds with a constant independent of M and ρ . Finally, letting $M \rightarrow \infty$ and $\rho \rightarrow 0$ we get (6.8) for all $f \in L^p(\mu)$ and $g \in L^q(\mu)$. \square

7. PROOF OF THE MAIN THEOREM

We are ready to prove our main theorem.

Proof of Theorem 1.2. Fix a $(2, b)$ -doubling cube Q_0 with t -small boundary, and set $\sigma = \mu \lfloor Q_0$. Now, the measure σ is of order m , $\sigma(\mathbb{R}^n \setminus Q_0) = 0$ and for $t_0 := tb$ we have

$$\sigma(\{x \in Q_0 : d(x, \partial Q_0) \leq \lambda \ell(Q_0)\}) \leq t_0 \lambda \sigma(Q_0)$$

for all $\lambda > 0$. Notice that $b_i \in L^\infty(\sigma)$, $i = 1, 2, 3$, and

$$|\langle b_i \rangle_Q^\sigma| = |\langle b_i \rangle_Q^\mu| \gtrsim 1 \quad \text{for all cubes } Q \subset Q_0.$$

Moreover, for all 1-Lipschitz functions $\Phi : \mathbb{R}^n \rightarrow [0, \infty)$, truncation parameters $\delta > 0$ and for all cubes Q satisfying $5Q \subset Q_0$ we have

$$|\langle T_{\sigma,\Phi,\delta}(1_Q b_1, 1_Q b_2), 1_Q b_3 \rangle_\sigma| = |\langle T_{\mu,\Phi,\delta}(1_Q b_1, 1_Q b_2), 1_Q b_3 \rangle_\mu| \lesssim \mu(5Q) = \sigma(5Q).$$

Corollary 2.3 gives a set H so that $\sigma(H) = \mu(H) \leq \mu(Q_0)/2 = \sigma(Q_0)/2$ and

$$\int_{Q_0 \setminus H} [S_{\sigma,\#}(b, b')]^{s/2} d\sigma = \int_{Q_0 \setminus H} [S_{\mu,\#}(1_Q b, 1_Q b')]^{s/2} d\mu \lesssim \mu(Q_0) = \sigma(Q_0)$$

for all the choices $(S, b, b') \in \{(T, b_1, b_2), (T^{1*}, b_3, b_2), (T^{2*}, b_1, b_3)\}$.

We are now in the position to use Theorem 4.2 to find a set $G \subset Q_0$ so that $\mu(G) = \sigma(G) \sim \sigma(Q_0) = \mu(Q_0)$ and uniformly over $\varepsilon > 0$ it holds that

$$\|T_{\mu \lfloor G, \varepsilon}\|_{L^4(\mu \lfloor G) \times L^4(\mu \lfloor G) \rightarrow L^2(\mu \lfloor G)} = \|T_{\sigma \lfloor G, \varepsilon}\|_{L^4(\sigma \lfloor G) \times L^4(\sigma \lfloor G) \rightarrow L^2(\sigma \lfloor G)} \lesssim 1.$$

It follows from Proposition 5.16 that

$$\|T_{\#}\|_{M(\mathbb{R}^n) \times M(\mathbb{R}^n) \rightarrow L^{1/2, \infty}(\mu \lfloor G)} \lesssim 1.$$

Since Q_0 was an arbitrary $(2, b)$ -doubling cube with t -small boundary, the good lambda method gives that for all $1 < p, q < \infty$ and $1/2 < r < \infty$ satisfying $1/p + 1/q = 1/r$ we have that

$$\|T_{\mu,\#}\|_{L^p(\mu) \times L^q(\mu) \rightarrow L^r(\mu)} \lesssim 1.$$

Therefore, we are done. \square

8. BRIEFLY ABOUT SQUARE FUNCTIONS

There is a natural class of bilinear square functions for which the above theory can also be developed. Let us formulate the setting and a boundedness theorem for these, and very briefly discuss the proof – and why it is significantly simpler than the Calderón–Zygmund case from above.

For each $t > 0$ and for two complex measures ν_1 and ν_2 on \mathbb{R}^n we define

$$\theta_t(\nu_1, \nu_2)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} s_t(x, y, z) d\nu_1(y) d\nu_2(z), \quad x \in \mathbb{R}^n.$$

The above integral converges absolutely if $\nu_1, \nu_2 \in M(\mathbb{R}^n)$ and the kernel satisfies (8.1) from below. For some $m, \alpha > 0$ the kernels $s_t: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ are assumed to satisfy the size condition

$$(8.1) \quad |s_t(x, y, z)| \lesssim \frac{t^{2\alpha}}{(t + |x - y|)^{m+\alpha}(t + |x - z|)^{m+\alpha}},$$

the x -Hölder condition

$$(8.2) \quad |s_t(x, y, z) - s_t(x', y, z)| \lesssim \frac{t^\alpha |x - x'|^\alpha}{(t + |x - y|)^{m+\alpha}(t + |x - z|)^{m+\alpha}}$$

whenever $|x - x'| < t/2$, the y -Hölder condition

$$(8.3) \quad |s_t(x, y, z) - s_t(x, y', z)| \lesssim \frac{t^\alpha |y - y'|^\alpha}{(t + |x - y|)^{m+\alpha}(t + |x - z|)^{m+\alpha}}$$

whenever $|y - y'| < t/2$, and the z -Hölder condition

$$(8.4) \quad |s_t(x, y, z) - s_t(x, y, z')| \lesssim \frac{t^\alpha |z - z'|^\alpha}{(t + |x - y|)^{m+\alpha}(t + |x - z|)^{m+\alpha}}$$

whenever $|z - z'| < t/2$. The bilinear vertical square function is defined by setting

$$BV(\nu_1, \nu_2)(x) = \left(\int_0^\infty |\theta_t(\nu_1, \nu_2)(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad x \in \mathbb{R}^n.$$

For a Borel measure μ of order m we set for $f, g \in \bigcup_{p \in [1, \infty]} L^p(\mu)$ and $x \in \mathbb{R}^n$ that

$$\theta_t^\mu(f, g)(x) = \theta_t(f d\mu, g d\mu)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} s_t(x, y, z) f(y) g(z) d\mu(y) d\mu(z)$$

and

$$BV_\mu(f, g)(x) = BV(f d\mu, g d\mu)(x) = \left(\int_0^\infty |\theta_t^\mu(f, g)(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

The above definitions make sense also when μ is finite. We use the notation $BV_\mu^A(f, g)(x)$ or $BV^A(\nu_1, \nu_2)(x)$ to mean that the integration \int_0^∞ is replaced with \int_0^A for some constant A .

For simplicity, we only state a $T1$ theorem, instead of a Tb theorem, in the square function setting.

8.5. Theorem. *Let μ be a measure of order m on \mathbb{R}^n , and $C_0 < \infty$, $\delta_0 < 1$ and $l > 0$ be given constants. Let $\beta > 0$ and C_1 be large enough (depending only on n). Suppose that for every $(2, \beta)$ -doubling cube $Q \subset \mathbb{R}^n$ with C_1 -small boundary there exists $H_Q \subset \mathbb{R}^n$ such that $\mu(H_Q) \leq \delta_0 \mu(Q)$ and*

$$(8.6) \quad \sup_{\lambda > 0} \lambda^l \mu(\{x \in Q \setminus H_Q : BV_\mu^{\ell(Q)}(1_Q, 1_Q)(x) > \lambda\}) \leq C_0 \mu(Q).$$

Then $BV_\mu : L^p(\mu) \times L^q(\mu) \rightarrow L^r(\mu)$ boundedly for all $1 < p, q < \infty$ and $1/2 < r < \infty$ satisfying $1/r = 1/p + 1/q$.

To prove this theorem one can just prove a big piece Tb (or $T1$) and then apply it in conjunction with the bilinear good lambda method. That is, one can completely skip all the difficulties involving adapted Cotlar type inequalities and transferring testing conditions to maximal truncations – this is simply because no maximal truncations appear, which is related to the fact that

$$\left(\int_A^\infty |\theta_t^\mu(f, g)(x)|^2 \frac{dt}{t} \right)^{1/2} \leq \left(\int_0^\infty |\theta_t^\mu(f, g)(x)|^2 \frac{dt}{t} \right)^{1/2}$$

for $A \geq 0$. This is one of the main things why the square function setting is easier, and this aspect would be greatly amplified in the context of local Tb theorems (which we do not consider in this paper). In addition, it is also the case that it is much simpler to prove the corresponding big piece $T1$. There are multiple reasons for this, but they include at least the following:

- Suppression is much easier and does not involve the sophisticated Lipschitz suppression;
- The paraproduct is simpler and the overall dyadic structure is simpler since one can use just one dyadic grid;
- Probabilistic arguments are easier since good functions are not needed (only good Whitney regions);
- The diagonal is trivial;
- There are less symmetries since no adjoints appear.

If one desires to read this much more approachable argument, one can find the full details in the first (v1) arXiv version of the current paper.

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